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THE ZETA-FUNCTION OF RIEMANN

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THE ZETA-FUNCTION OF RIEMANN

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PREFACE

SOME years ago Prof. H. Bohr and Prof. J. E. Littlewood began to write a Cambridge tract on prime-number theory and the zeta-function of Riemann. Their work was never finished, but they prepared a manuscript of considerable size which still exists. The fact that it is now somewhat out-of-date is due in no small part to the subsequent researches of Bohr and Littlewood themselves.

The present work deals only with the theory of the zeta-function itself, without regard to its applications in the theory of numbers. These applications will form the subject of a companion volume by Mr A. E. Ingham. No attempt is made to explain the connection between the two subjects. We give only a brief sketch of the better-known properties of the function, which the reader will probably have learnt elsewhere. Apart from this the work is complete in itself.

The early part of Chapter IV is taken from the Bohr-Littlewood MS. with but slight changes. In several places I have made use of Prof. Littlewood's lecture notes, which he has kindly placed at my disposal. I have also to thank Dr T. Estermann and Prof. G. B. Jeffery for reading the manuscript, and Miss M. L. Cartwright and Prof. G. H. Hardy for reading the proof-sheets. A large number of corrections and suggestions are due to them.

E. C. T.

LIVERPOOL
May, 1930

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INTRODUCTION

This tract is intended for readers who already have some knowledge of the zeta-function and its rôle in the analytical theory of numbers; but for the sake of completeness we give a brief sketch of its elementary properties.

The function is defined by the Dirichlet series

$$(1) \qquad \zeta(s) = \sum_{n=1} n^{-s},$$

where $s = \sigma + it$, and the series is convergent, and the function analytic, for $\sigma > 1$.

We have also

$$(2) \qquad \zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \quad (\sigma > 1),$$

where p runs through all prime numbers. If we expand each factor in powers of p^{-s} and multiply formally, the result follows from the fact that each integer n can be expressed as a product of prime powers p^m in just one way. The rigorous proof is easily constructed by taking first a finite number of factors.

From the convergence of the product (2) *we deduce that $\zeta(s)$ has no zeros for $\sigma > 1$.*

We derive from (2) the formulae

$$(2') \qquad \log \zeta(s) = \sum_{p} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^s},$$

$$(2'') \qquad \frac{\zeta'(s)}{\zeta(s)} = -\sum_{p} \sum_{m=1}^{\infty} \frac{\log p}{p^{ms}} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n) = \log p$, if n is a power of the prime p, and otherwise $\Lambda(n) = 0$; and $\Lambda_1(n) = \Lambda(n)/\log n$.

A third representation of the function is

$$(3) \qquad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \quad (\sigma > 1).$$

For

$$n^{-s}\Gamma(s) = \int_0^\infty x^{s-1} e^{-nx}\, dx,$$

and we obtain the result on summing with respect to n and inverting the order of summation and integration.

We next replace the integral by the contour integral

$$(4) \qquad \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z-1}\,dz,$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction (but excludes the points $\pm 2i\pi$, $\pm 4i\pi$, ...) and returns to its starting point. The numerator of the integrand is interpreted as being $\exp\{(s-1)\log(-z)\}$, where the logarithm is real on the negative real axis. This formula is deduced from (3) in the case $\sigma > 1$ by shrinking the contour C into the real axis described twice, and taking account of the different value of the logarithm on the two parts.

Now this integral is uniformly convergent in any finite region, and so represents an integral function of s. This enables us to continue $\zeta(s)$ over the whole plane. *Hence $\zeta(s)$ is regular for all values of s except for a simple pole at $s=1$, with residue* 1. For the poles of $\Gamma(1-s)$ are at $s=1, 2, \ldots$, and we already know that $\zeta(s)$ is regular at all these points except $s=1$. Also, if s is an integer, the integral can be evaluated by the theorem of residues. Since

$$\frac{z}{e^z-1} = 1 - \tfrac{1}{2}z + B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} + \ldots,$$

we find the following values of $\zeta(s)$:

$$(5) \quad \zeta(0) = -\tfrac{1}{2}, \quad \zeta(-2m) = 0, \quad \zeta(1-2m) = \frac{(-1)^m B_m}{2m} \ (m = 1, 2, \ldots).$$

We next expand the contour C into a position C_n in which it includes the poles of the integrand at $\pm 2i\pi, \ldots \pm 2ni\pi$. The sum of the residues at these points is found to be

$$2\sum_{m=1}^{n}(2m\pi)^{s-1}\sin\tfrac{1}{2}s\pi.$$

If $\sigma < 0$ we can make $n \to \infty$; the integral round C_n tends to zero, and we obtain the functional equation*

$$(6) \qquad \zeta(s) = 2^s\pi^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)\,\zeta(1-s)$$

or

$$(6') \qquad \zeta(1-s) = 2^{1-s}\pi^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s)\,\zeta(s).$$

This holds (by continuation) for all values of s. We deduce from this that *$\zeta(s)$ has no zeros in the half-plane $\sigma < 0$ except at $s = -2, -4, \ldots$.*

* For alternative proofs see *Handbuch*, i, 281–299, Hardy (2), (3), Mordell (1), (2). *Handbuch* is used throughout as an abbreviation of E. Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, 1909. The numbers in brackets refer to the bibliography at the end.

Writing

$$\xi(s) = \tfrac{1}{2} s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s), \tag{7}$$

the functional equation takes the simple form

$$\xi(s) = \xi(1-s). \tag{6''}$$

Also if $\Xi(z) = \xi(\frac{1}{2} + iz)$, then $\Xi(z)$ *is an even function of* z.

What is the order* of $\Xi(z)$, or $\xi(s)$? By (6″) it is sufficient to consider the half-plane $\sigma \geqq \frac{1}{2}$. Now we easily verify the formula

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{s-1}\frac{1}{N^{s-1}} - s\int_N^\infty \frac{u-[u]}{u^{s+1}}\,du \quad (\sigma > 0) \tag{8}$$

and deduce that†, for $\sigma \geqq \delta\,(0 < \delta < 1)$, $|t| \geqq 1$,

$$\zeta(s) = O(N^{1-\delta}) + O(tN^{-\delta}) = O(t^{1-\delta}) \tag{9}$$

on taking $N = [t]$. Since also

$$|\Gamma(\tfrac{1}{2}s)| \leqq |\Gamma(\tfrac{1}{2}\sigma)| = O(e^{A\sigma\log\sigma}),$$

it follows from (7) that $\xi(s)$ is of order 1 at most. By considering real positive values of s it is easily seen that *the order is exactly* 1.

It follows that $\Xi(\sqrt{z})$ is an integral function of order $\frac{1}{2}$, and so has an infinity of zeros. From this we deduce that $\zeta(s)$ has an infinity of zeros other than the real ones already observed. These zeros must be complex and lie in the strip $0 \leqq \sigma \leqq 1$.

We can now write

$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(\frac{1}{2}s+1)}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \tag{10}$$

where ρ runs through the complex zeros of $\zeta(s)$. For by Hadamard's factorization theorem‡

$$\xi(s) = \xi(0)\,e^{b_0 s}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}.$$

Here $\xi(0) = -\zeta(0) = \frac{1}{2}$, and

$$b = b_0 + \tfrac{1}{2}\log\pi = \log 2\pi - 1 - \tfrac{1}{2}\gamma, \tag{11}$$

where γ is Euler's constant.

The proof of (11) is rather complicated. In the first place we have

$$\lim_{s\to 1}\left\{\zeta(s) - \frac{1}{s-1}\right\} = \gamma. \tag{12}$$

* See G. Valiron's *Lectures on the General Theory of Integral Functions*, 1923, Ch. II, § 5.

† Throughout the tract A denotes a positive absolute constant, not necessarily the same at each occurrence. For each n, A_n always denotes the same constant. $A(\delta)$ denotes a constant depending on δ. $f(t) = O\{\phi(t)\}$ means that $|f(t)| < A|\phi(t)|$ for $t > A$. $f(t) = \Omega\{\psi(t)\}$ means that $|f(t)| > A|\psi(t)|$ for some indefinitely large values of t. If, however, ϕ or ψ depends on δ, the constant implied in O or Ω is $A(\delta)$. ϵ always denotes an arbitrarily small positive number.

‡ See Valiron's *Integral Functions*, Ch. III, § 4.

For by (8)

$$\zeta(s)-\frac{1}{s-1}=\left\{\sum_{n=1}^{N}\frac{1}{n}-\int_1^N\frac{du}{u}\right\}+\sum_{n=1}^{N}\left(\frac{1}{n^s}-\frac{1}{n}\right)$$
$$-\int_1^N\left(\frac{1}{u^s}-\frac{1}{u}\right)du-s\int_N^\infty\frac{u-[u]}{u^{s+1}}du \quad (\sigma>1).$$

We can choose N so large that the last term is as small as we please, and at the same time so that the first term differs from γ by as little as we please, and this independently of σ. Having fixed N, the remaining terms tend to zero with $s-1$.

This proves (12). We can now calculate $\zeta'(0)$ by differentiating (6) logarithmically, making $s\to 0$, and using (5), (12) and the fact that $\Gamma'(1)=-\gamma$. We obtain

$$(13)\qquad \zeta'(0)=-\tfrac{1}{2}\log 2\pi.$$

Finally $b_0=\xi'(0)/\xi(0)$, and (11) follows from (13).

As to the zeros of $\zeta(s)$, it is known that *there are none on the line* $\sigma=1$. We deduce from (2′) that

$$(14)\qquad \zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)|$$
$$=\exp\left\{\Sigma\Sigma\frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}}\right\}.$$

Since

$$(15)\qquad 3+4\cos\phi+\cos 2\phi=2(1+\cos\phi)^2\geqq 0,$$

every term in the exponent on the right of (14) is positive, and hence the left-hand side is not less than 1. Putting $\sigma=1+\epsilon$ ($\epsilon<1$), and noting that

$$(16)\qquad \zeta(1+\epsilon)=\Sigma n^{-1-\epsilon}<1+\int_1^\infty u^{-1-\epsilon}\,du=1+\frac{1}{\epsilon}<\frac{2}{\epsilon},$$

we obtain
$$\frac{|\zeta(1+\epsilon+it)|}{\epsilon}>\frac{1}{2\epsilon^{\frac14}\,|\zeta(1+\epsilon+2it)|^{\frac14}}.$$

Since $\zeta(s)$ is analytic, the left-hand side would tend to $|\zeta'(1+it)|$ as $\epsilon\to 0$, if $1+it$ were a zero of $\zeta(s)$. But the right-hand side tends to infinity. Hence $\zeta(s)$ cannot have a zero on $\sigma=1$.

The number* $N(T)$ of zeros of $\zeta(s)$ between $t=0$ and $t=T$ is given approximately by the formula

$$(17)\qquad N(T)=\frac{1}{2\pi}T\log T-\frac{1+\log 2\pi}{2\pi}T+O(\log T).$$

This number is the same for $\xi(s)$ as for $\zeta(s)$, and is half the number of zeros in the rectangle $-1<\sigma<2$, $-T<t<T$.

* *Handbuch*, 368–372, Backlund (2), (3).

Hence, if T is not the ordinate of a zero, $4\pi N(T)$ is equal to the variation of am $\xi(s)$ round the perimeter of this rectangle. Now $\xi(s)$ is real for $t=0$, and also for $\sigma=\frac{1}{2}$, so that the variation round the whole rectangle is four times the variation round a quarter of it (starting from $s=2$). Hence $\pi N(T)=\Delta \text{ am } \xi(s)$, where Δ denotes the variation from 2 to $2+iT$, and thence to $\frac{1}{2}+iT$, along straight lines. Expressing $\xi(s)$ again in terms of $\zeta(s)$, and using the asymptotic formula* for $\Gamma(s)$, we obtain

$$(18)\quad N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+\frac{1}{\pi}\Delta \text{ am } \zeta(s)+O\left(\frac{1}{T}\right).$$

The real difficulty of the proof lies in proving that $\Delta \text{ am } \zeta(s)=O(\log T)$.

In the first place, $\mathbf{R}\{\zeta(s)\}$ does not vanish on $\sigma=\frac{1}{2}$, since

$$\mathbf{R}\,\zeta(2+it) \geqq 1-\sum_{n=2}^{\infty} n^{-2}>0.$$

Hence the variation of am $\zeta(s)$ from 2 to $2+iT$ is less than $\frac{1}{2}\pi$.

Secondly, if $\mathbf{R}\{\zeta(s)\}$ vanishes q times between $2+iT$ and $\frac{1}{2}+iT$, this interval is divided into $q+1$ parts, throughout each of which $\mathbf{R}\{\zeta(s)\} \geqq 0$ or $\mathbf{R}\{\zeta(s)\} \leqq 0$. Hence in each part the variation of am $\zeta(s)$ does not exceed π, and consequently $|\Delta \text{ am } \zeta(s)| \leqq (q+\frac{3}{2})\pi$.

Now q is the number of zeros of $f(z)=\frac{1}{2}\{\zeta(z+iT)+\zeta(z-iT)\}$ for $\mathbf{I}(z)=0$, $\frac{1}{2} \leqq \mathbf{R}(z) \leqq 2$, and so $q \leqq n$, where n is the number of zeros of $f(z)$ in $|z-2| \leqq \frac{3}{2}$. By Jensen's theorem†

$$n<A\left\{\int_0^{2\pi} \log|f(2+re^{i\theta})|\,d\theta-2\pi\log|f(2)|\right\},$$

where $\frac{3}{2}<r<2$; and by (9) the right-hand side is less than $A\log T$. This proves (17).

An immediate consequence of (17) is‡

$$(19)\qquad N(T+1)-N(T)=O(\log T).$$

This however follows directly from Jensen's theorem in the same way as the above result for $f(z)$.

We have also shewn incidentally that

$$(20)\qquad \text{am } \zeta(s)=O(\log t)\quad (\sigma \geqq \tfrac{1}{2}).$$

* Whittaker and Watson, *Modern Analysis* (ed. 4, 1927), § 13·6.

† Valiron's *Integral Functions*, Ch. III, § 1.

‡ *Handbuch*, 337.

CHAPTER I

THE ASYMPTOTIC BEHAVIOUR OF $\zeta(s)$

1.0. Introduction. The fundamental problem which emerges from the attempt to determine the distribution of the prime numbers is: where are the zeros of the zeta-function? We also encounter the problem of the asymptotic behaviour of $\zeta(s)$, as $t \to \infty$, for given values of σ. The two problems are closely connected, and we cannot entirely separate them. But we find it convenient to deal with the second question first as far as possible. Accordingly all the theorems in our first chapter are on inequalities, except Theorem 12, on the zeros, which falls into place naturally here.

We take a certain value (or range of values) of σ, and ask, how large are the largest values of $|\zeta(s)|$, and how small are its smallest values? Actually we always put the second question in the form, how large are the largest values of $|1/\zeta(s)|$? More precisely, we seek to determine monotonic functions $\phi(t)$ and $\psi(t)$ such that

$$\zeta(s) = O\{\phi(t)\}, \quad \zeta(s) = \Omega\{\psi(t)\},$$

and then consider the same problem for $1/\zeta(s)$. If $\phi(t) = \psi(t)$, the problem takes the form of determining the best possible values of the constants.

1.1. $\zeta(s)$ *in the half-plane* $\sigma > 1$.

1.11. In the half-plane $\sigma > 1$, where $\zeta(s)$ is represented by an absolutely convergent Dirichlet series, its main features are known.

Theorem 1. *If* $\sigma > 1$, *then*

$$|\zeta(s)| \leqq \zeta(\sigma)$$

for all values of t, *while*

$$|\zeta(s)| \geqq (1-\epsilon)\zeta(\sigma)$$

for some indefinitely large values of t.

We have* $$|\zeta(s)| = |\Sigma n^{-s}| \leqq \Sigma n^{-\sigma} = \zeta(\sigma).$$

So the whole difficulty lies in the second part.

1.12. We use the following well-known theorem on Diophantine approximation.

Theorem of Dirichlet. *Given* N *real numbers* $a_1, a_2, \ldots a_N$, *a positive integer* q, *and a positive number* τ, *we can find a number* t *in the range*

$$\tau \leqq t \leqq \tau q^N,$$

and integers $x_1, x_2, \ldots x_N$, *such that*

$$|ta_n - x_n| \leqq 1/q \quad (n = 1, 2, \ldots N).$$

* The limits of summation are always $(1, \infty)$ unless otherwise stated.

The proof is based on an argument which was introduced and employed extensively by Dirichlet. This argument, in its simplest form, is that, if there are $m+1$ points in m regions, there must be at least one region which contains at least two points.

Consider the N-dimensional unit cube with a vertex at the origin and edges along the coordinate axes. Divide each edge into q equal parts, and thus the cube into q^N equal compartments. Consider the q^N+1 points, in the cube, congruent (mod 1) to the points $(ua_1, ua_2, \dots ua_N)$, where $u=0, \tau, \dots, \tau q^N$. At least two of these points must lie in the same compartment. If these two points correspond to $u=u_1$, $u=u_2$, $(u_1<u_2)$, then $t=u_2-u_1$ satisfies the requirements of the theorem.

1.13. We now apply this to $\zeta(s)$. For all values of N

$$\zeta(s)=\sum_{n=1}^{N} n^{-\sigma}e^{-it\log n}+\sum_{n=N+1}^{\infty} n^{-\sigma-it},$$

and hence (the modulus of the first sum being not less than its real part)

$$|\zeta(s)| \geqq \sum_{n=1}^{N} n^{-\sigma}\cos(t\log n)-\sum_{n=N+1}^{\infty} n^{-\sigma}. \tag{1}$$

By Dirichlet's theorem there is a number t $(\tau \leqq t \leqq \tau q^N)$, and integers $x_1, \dots, x_N$, such that, for given N and q,

$$|t\log n/2\pi - x_n| < 1/q \quad (n=1,2,\dots N).$$

Hence $\cos(t\log n) > \cos(2\pi/q)$ for these values of n, and so

$$\sum_{n=1}^{N} n^{-\sigma}\cos(t\log n) > \cos(2\pi/q)\sum_{n=1}^{N} n^{-\sigma} > \cos(2\pi/q)\,\zeta(\sigma)-\sum_{N+1}^{\infty} n^{-\sigma};$$

and hence by (1)

$$|\zeta(s)| \geqq \cos(2\pi/q)\,\zeta(\sigma)-2\sum_{N+1}^{\infty} n^{-\sigma}. \tag{2}$$

Now

$$\zeta(\sigma)=\sum n^{-\sigma} > \int_1^{\infty} u^{-\sigma}\,du=\frac{1}{\sigma-1}, \tag{3}$$

$$\sum_{N+1}^{\infty} n^{-\sigma} < \int_N^{\infty} u^{-\sigma}\,du=\frac{N^{1-\sigma}}{\sigma-1}.$$

Hence

$$|\zeta(s)| \geqq \{\cos(2\pi/q)-2N^{1-\sigma}\}\,\zeta(\sigma), \tag{4}$$

and the result follows if q and N are large enough.

1.14. A consequence of Theorem 1 is that $\zeta(s)$, though bounded on any fixed line $\sigma=\sigma_0>1$, is not bounded in the open region $\sigma>1$, $t>t_0>1$.

This follows at once from the fact that the upper bound $\zeta(\sigma)$ of $\zeta(s)$ tends to infinity as $\sigma \to 1$. But a little additional argument gives the following more precise and more interesting result*.

THEOREM 2. *However large t_0 may be, there are values of s in the region $\sigma > 1$, $t > t_0$, for which*

(1) $$|\zeta(s)| > A \log\log t.$$

Take $\tau = 1$ and $q = 6$ in § 1.13. Then, by 1.13 (3) and (4),

(2) $$|\zeta(s)| \geqq \{\tfrac{1}{2} - 2N^{1-\sigma}\}/(\sigma - 1)$$

for a value of t between 1 and 6^N. We choose N to be the integer next above $8^{1/(\sigma-1)}$. Then

(3) $$|\zeta(s)| \geqq \frac{1}{4(\sigma-1)} > \frac{\log(N-1)}{4\log 8} > A \log N$$

for a value of t such that

(4) $$N > A \log t.$$

The required inequality (1) follows from (3) and (4). It remains only to observe that the value of t in question must be greater than any assigned t_0, if $\sigma - 1$ is sufficiently small; otherwise it would follow from (3) that $\zeta(s)$ was unbounded in the region $\sigma > 1$, $1 < t < t_0$; and we know that $\zeta(s)$ is bounded in any such region.

1.15. There remains the question as to how large the constant A of Theorem 2 may be. *It has been proved that Theorem 2 is true if A has any value less than e^{γ}, where γ is Euler's constant*†. This particular constant arises from the formula‡

$$\prod_{p<x}\left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}.$$

1.16. It is plain that the above argument may be applied to all Dirichlet series, with coefficients of fixed sign, which are not absolutely convergent on their line of convergence. For example, the series for $\log \zeta(s)$ and its differential coefficients are of this type. The result for $\log \zeta(s)$ is however a corollary of that for $\zeta(s)$, which gives at once

$$|\log \zeta(s)| > \log\log\log t - A$$

for some indefinitely large values of t in $\sigma > 1$. For the nth differential coefficient of $\log \zeta(s)$ the result is that

* Bohr and Landau (1). † Littlewood (5), Titchmarsh (4).
‡ *Handbuch*, §§ 28 and 36, Hardy (4).

$$\left|\left(\frac{d}{ds}\right)^n \log \zeta(s)\right| > A(n)(\log\log t)^n$$

for some indefinitely large values of t in $\sigma > 1$.

1.21. We now turn to the corresponding problem for $1/\zeta(s)$.

THEOREM 3. *If $\sigma > 1$, then*

$$\left|\frac{1}{\zeta(s)}\right| \leqq \frac{\zeta(\sigma)}{\zeta(2\sigma)}$$

for all values of t, while

$$\left|\frac{1}{\zeta(s)}\right| \geqq (1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)}$$

for some indefinitely large values of t.

As before, the first part is almost immediate. We have

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum \frac{\mu(n)}{n^s},$$

where $\quad \mu(1) = 1, \quad \mu(n) = (-1)^k$

if n is the product of k different primes, and otherwise $\mu(n) = 0$. Hence

$$\left|\frac{1}{\zeta(s)}\right| \leqq \sum \frac{|\mu(n)|}{n^\sigma} = \prod_p \left(\frac{1-p^{-2\sigma}}{1-p^{-\sigma}}\right) = \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

1.22. We have now to prove the second part. In this problem Dirichlet's theorem fails to give any result, since the coefficients $\mu(n)$ are not all of one sign. We therefore leave Diophantine approximation for the moment, and adopt a different method* (which may also be used to prove Theorem 1). It depends on the fact that, if $\phi(t)$ is positive and continuous for $a \leqq t \leqq b$, then

$$\lim_{n\to\infty} \left[\frac{1}{b-a}\int_a^b \{\phi(t)\}^n\, dt\right]^{1/n} = \operatorname*{Max}_{a \leqq t \leqq b} \phi(t).$$

We can thus reduce the question of the maximum value of a function to that of the behaviour of integrals involving high powers of the function.

1.23. We also require the following mean value theorem for Dirichlet series. We write

$$M\{|f|^2\} = \lim_{T\to\infty} \frac{1}{2T}\int_T^T |f(\sigma+it)|^2\, dt.$$

If $f(s) = \Sigma a_n n^{-s}$ is absolutely convergent for $\sigma = \sigma_0$, then, for $\sigma = \sigma_0$,

$$M\{|f|^2\} = \Sigma\, |a_n|^2 n^{-2\sigma_0}.$$

* Bohr and Landau (7).

We have

$$|f(\sigma_0+it)|^2=\Sigma a_m m^{-\sigma_0-it}\Sigma \bar{a}_n n^{-\sigma_0+it}$$
$$=\Sigma\,|\,a_n\,|^2 n^{-2\sigma_0}+\Sigma\Sigma' a_m\bar{a}_n(mn)^{-\sigma_0}(n/m)^{it},$$

the accent indicating that terms with $m=n$ are omitted. Hence

$$\frac{1}{2T}\int_T^T|f(\sigma_0+it)|^2dt=\Sigma\frac{|\,a_n\,|^2}{n^{2\sigma_0}}+\Sigma\Sigma'\frac{a_m\bar{a}_n}{(mn)^{\sigma_0}}\frac{1}{2T}\int_{-T}^T\left(\frac{n}{m}\right)^{it}dt.$$

The last double series converges uniformly with respect to T, since the absolute value of its general term is not greater than $|\,a_m a_n|\,(mn)^{-\sigma_0}$; and each term tends to zero as $T\to\infty$. Hence the sum of the series tends to zero, and the result follows.

1.24. We can now prove the second part of Theorem 3. We write

$$\frac{1}{\zeta(s)}=\prod_{n=1}^N\left(1-\frac{1}{p_n^s}\right)\eta_N(s),$$

and then raise each side to the kth power. We thus express $\{\zeta(s)\}^{-k}$ as the product of $N+1$ absolutely convergent Dirichlet series, the 'm' of every term $a_m m^{-s}$ with non-zero coefficient in any one of them being prime to the 'n' of any non-zero $a_n n^{-s}$ in any other. It is easily seen from this and the theorem of § 1.23 that

$$M\{|\zeta^{-2k}(s)|\}=\prod_{n=1}^N M\{|(1-p_n^{-s})^{2k}|\}\,M\{|\eta_N^{2k}(s)|\}.$$

Now for every p

$$M\{|1-p^{-s}|^{2k}\}=\frac{\log p}{2\pi}\int_0^{2\pi/\log p}|1-p^{-s}|^{2k}dt,$$

since the integrand is periodic with period $2\pi/\log p$; and

$$M\{|\eta_N^{2k}(s)|\}\geqq 1,$$

since the Dirichlet series for $\eta_N^k(s)$ begins with $1+\ldots$. Hence

$$M\left\{\left|\frac{1}{\zeta(s)}\right|^{2k}\right\}\geqq\prod_{n=1}^N\frac{\log p_n}{2\pi}\int_0^{2\pi/\log p_n}|1-p_n^{-s}|^{2k}dt.$$

Now

$$\lim_{k\to\infty}\left\{\frac{\log p}{2\pi}\int_0^{2\pi/\log p}|1-p^{-s}|^{2k}dt\right\}^{1/2k}=\operatorname*{Max}_{0\leqq t\leqq 2\pi/\log p}|1-p^{-s}|=1+p^{-\sigma}.$$

Hence
$$\varliminf_{k\to\infty}\,[M\{|\zeta(s)|^{-2k}\}]^{1/2k}\geqq\prod_{n=1}^N(1+p_n^{-\sigma}).$$

Since this is true for all values of N, we can replace the right-hand side by its limit as $N\to\infty$, viz. $\zeta(\sigma)/\zeta(2\sigma)$. Hence to any ϵ corresponds a k such that

$$\left[M\left\{\left|\frac{1}{\zeta(s)}\right|^{2k}\right\}\right]^{1/2k}>(1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)},$$

and therefore there are indefinitely large values of t for which

$$\left|\frac{1}{\zeta(s)}\right| \geqq (1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

This proves the theorem.

1.25. THEOREM 4*. *The function* $1/\zeta(s)$ *is not bounded in the open region* $\sigma > 1$, $t > t_0 > 0$.

This follows at once from Theorem 3 and the fact that as $\sigma \to 1$

$$\zeta(\sigma)/\zeta(2\sigma) \to \infty.$$

There is another proof of this result, depending on Diophantine approximation. We have already noticed that Dirichlet's theorem is not available in this case; but there is another theorem rather like it which we can use.

1.26. KRONECKER'S THEOREM †. *If* $a_1, a_2, \ldots a_N$ *are linearly independent, and* $b_1, b_2, \ldots b_N$ *are any real numbers, then, given* q, *we can find a number* t *and integers* $x_1, x_2, \ldots x_N$ *such that*

$$|a_n t - b_n - x_n| < 1/q \quad (n = 1, 2, \ldots N).$$

If all the numbers b_n are zero, the result is included in Dirichlet's theorem. The general theorem requires that the numbers a_n should be linearly independent, i.e. that there should be no relation of the form

$$r_1 a_1 + r_2 a_2 + \ldots + r_N a_N = 0,$$

where $r_1, r_2, \ldots$ are integers. Also it assigns no upper bound for the number t such as the q^N of Dirichlet's theorem. The consequence of this is that from Kronecker's theorem we can deduce only unboundness, and not a definite order-result such as Theorem 2.

1.27. To prove Theorem 4 we write

$$\log\zeta(s) = \sum_p p^{-s} + \phi(s),$$

where $\phi(s)$ is bounded for $\sigma > 1$. Now

$$\mathbf{R}\left(\sum_p p^{-s}\right) = \sum \cos(t\log p_n)p_n^{-\sigma} \leqq \sum_{n=1}^{N}\cos(t\log p_n)p_n^{-\sigma} + \sum_{n=N+1}^{\infty} p_n^{-\sigma}.$$

Also the numbers $\log p_n$ *are linearly independent.* For it follows from the theorem, that an integer can be expressed as a product of prime factors in one way only, that there can be no relation of the form

$$p_1^{r_1} p_2^{r_2} \ldots p_N^{r_N} = 1,$$

and therefore no relation of the form

$$r_1 \log p_1 + \ldots + r_N \log p_N = 0.$$

* Bohr and Landau (7).

† Various proofs of Kronecker's theorem are known. See, e.g., F. Lettenmeyer, *Proc. London Math. Soc.* (2), 21 (1922), 306–314; H. Bohr, *ibid.* 315–316. We give Bohr's proof in an appendix.

Hence also the numbers $\log p_n/2\pi$ are linearly independent. It follows therefore from Kronecker's theorem that we can find a number t and integers $x_1, \dots x_N$ such that

$$|t \log p_n/2\pi - \tfrac{1}{2} - x_n| < \tfrac{1}{6} \quad (n = 1, 2, \dots, N),$$

or

$$|t \log p_n - \pi - 2\pi x_n| < \tfrac{1}{3}\pi \quad (n = 1, 2, \dots, N).$$

Hence, for these values of n,

$$\cos(t \log p_n) = -\cos(t \log p_n - \pi - 2\pi x_n) < -\cos\tfrac{1}{3}\pi = -\tfrac{1}{2},$$

and hence

$$\mathbf{R}\,(\sum_p p^{-s}) < -\tfrac{1}{2} \sum_{n=1}^{N} p_n^{-\sigma} + \sum_{n=N+1}^{\infty} p_n^{-\sigma}.$$

Since Σp_n^{-1} is divergent, we can, if H is any assigned positive number, choose σ so near to 1 that $\Sigma p_n^{-\sigma} > H$. Having fixed σ, we can choose N so large that

$$\sum_{n=1}^{N} p_n^{-\sigma} > \tfrac{3}{4}H, \quad \sum_{n=N+1}^{\infty} p_n^{-\sigma} < \tfrac{1}{4}H.$$

Then

$$\mathbf{R}\,(\sum_p p^{-s}) < -\tfrac{3}{8}H + \tfrac{1}{4}H = -\tfrac{1}{8}H.$$

Since H may be as large as we please, it follows that $\mathbf{R}\,(\sum_p p^{-s})$, and so $\log|\zeta(s)|$, takes arbitrarily large negative values. This proves the theorem.

1.28. We have already remarked that in the general case of Kronecker's theorem we can assign no upper bound to the number t; but Bohr and Landau* have proved the remarkable result that, in the particular case where $a_n = \log p_n/2\pi$, a value of t can be found which is, roughly, less than an upper bound not very different from q^N. This discovery enables us to prove a result analogous to Theorem 2.

THEOREM 5. *However large t_0 may be, there are values of s in the region $\sigma > 1$, $t > t_0$ for which*

$$|1/\zeta(s)| > A \log\log t.$$

1.3. *The function $\zeta(1 + it)$.*

1.31. THEOREM 6. *We have*

$$\zeta(s) + O(\log t)$$

uniformly in any region

$$1 - \frac{A}{\log t} \leqq \sigma \leqq 2, \quad t > t_0.$$

In particular

$$\zeta(1 + it) = O(\log t).$$

* Bohr and Landau (6), (7).

In the region considered, if $n \leqq t$,

$$|n^{-s}| = n^{-\sigma} \leqq \exp\left\{-\left(1-\frac{A}{\log t}\right)\log n\right\} \leqq \frac{e^A}{n}.$$

Hence, by (8) of the Introduction, with $N = [t]$,

$$\begin{aligned}\zeta(s) &= \sum_1^N O\left(\frac{1}{n}\right) + O\left(\frac{1}{s-1}\right) + O\left(t\int_N^\infty \frac{du}{u^{\sigma+1}}\right)\\ &= O(\log N) + O(1/t) + O(tN^{-\sigma})\\ &= O(\log t) + O(1) + O(1).\end{aligned}$$

This result will be replaced later by a slightly better one (Theorem 11) which, however, requires a far more difficult proof.

1.32. THEOREM 7*.

$$\zeta(1+it) = \Omega(\log\log t).$$

We have already seen that $\zeta(s)$ takes arbitrarily large values in the immediate neighbourhood of the line $\sigma = 1$, but our argument by itself does not tell us anything about the values on the line $\sigma = 1$. There is however a well-known theorem, due to Lindelöf, which enables us to deal with questions of precisely this kind.

LINDELÖF'S THEOREM†. *Suppose that a function $f(s)$ is regular and of the form $O(t^A)$ in the half-strip*

$$\sigma_1 \leqq \sigma \leqq \sigma_2, \quad t \geqq t_0 > 0;$$

and that $|f(s)| \leqq M$ on the whole boundary of the half-strip. Then $|f(s)| \leqq M$ throughout the half-strip.

Suppose now that the function $\zeta(1+it)$ were bounded; we know that $\zeta(2+it)$ is bounded; and we know from Theorem 6 that $\zeta(s) = O(t^A)$ in the half-strip

$$1 \leqq \sigma \leqq 2, \quad t \geqq t_0 > 0.$$

We should therefore conclude that $\zeta(s)$ was bounded in this half-strip, whereas we know from Theorem 1 that it is not. *Hence $\zeta(1+it)$ is unbounded.*

We can however use the more precise information of Theorem 2, concerning the asymptotic behaviour of $\zeta(s)$ in $\sigma > 1$, to obtain correspondingly more precise information concerning its behaviour on the line $\sigma = 1$.

* Bohr and Landau (1).

† A proof of this is given by Hardy and Riesz, *The General Theory of Dirichlet's Series*, p. 15.

In the first place it follows from Lindelöf's theorem *that if $f(s)$ is regular and of the form $O(t^A)$ in the half-strip,*

$$\sigma_1 \leqq \sigma \leqq \sigma_2, \quad t \geqq t_0,$$

and if $f(s) \to 0$ on both the lines $\sigma = \sigma_1$, $\sigma = \sigma_2$, as $t \to \infty$, then $f(s) \to 0$ uniformly in the half-strip.

The deduction, while easy, is not completely trivial. We consider the function

$$F(s) = f(s)\frac{s}{s+h}.$$

By choosing h sufficiently large we can ensure that $|F(s)| < \epsilon$ on the boundary, and so, by the main theorem, also throughout the half-strip. Hence

$$|f(s)| = |(1 + h/s)F(s)| < 2\epsilon$$

for $|s| > h$, which gives the desired result.

Now we see from Theorem 2 that

$$f(s) = \zeta(s)/\log\log s$$

does not tend to zero uniformly for $1 < \sigma < 2$ as $t \to \infty$; and $f(2+it) \to 0$. The above theorem therefore shews that $f(1+it)$ does not tend to zero, and this proves Theorem 7.

By using the result mentioned in § 1.15 we can prove further that

$$\overline{\lim}\,\frac{\zeta(1+it)}{\log\log t} \geqq e^{\gamma}.$$

1.33. The reader will see that there is a considerable gap between the O-result of Theorem 6 and the Ω-result of Theorem 7; and even Theorem 11 does little towards narrowing the gap. The real truth may of course lie anywhere in the gap. But we shall see later that the Ω-result is probably (i.e. if the Riemann hypothesis is true) the best possible, and that it is the O-result which falls short of the truth. A similar remark applies to all our O- and Ω-results. We are always more successful with Ω-theorems. This is perhaps not surprising, since an O-result is a statement about all large values of t, an Ω-result about some indefinitely large values only.

1.4. *The zeros of $\zeta(s)$ and the function $1/\zeta(1+it)$.*

1.41. Before we can deal with $1/\zeta(1+it)$ we have to prove that $\zeta(s)$ has no zeros in the immediate neighbourhood of the line $\sigma = 1$. For this purpose we use the following lemmas*.

* Landau (9).

LEMMA. *If $f(s)$ is regular, and*

$$|f(s)/f(s_0)| < e^M \quad (M > 1)$$

in the circle $|s - s_0| \leq r$, then

$$\left|\frac{f'(s)}{f(s)} - \Sigma \frac{1}{s-\rho}\right| < A\frac{M}{r} \quad (|s - s_0| \leq \tfrac{1}{4}r),$$

where ρ runs through the zeros of $f(s)$ in $|s - s_0| \leq \frac{1}{2}r$.

The function $g(s) = f(s)\,\Pi\,(s-\rho)^{-1}$ is regular for $|s - s_0| \leq r$, and not zero for $|s - s_0| \leq \frac{1}{2}r$. On $|s - s_0| = r$, $|s - \rho| \geq \frac{1}{2}r \geq |s_0 - \rho|$, so that

$$\left|\frac{g(s)}{g(s_0)}\right| = \left|\frac{f(s)}{f(s_0)}\,\Pi\left(\frac{s_0-\rho}{s-\rho}\right)\right| \leq \left|\frac{f(s)}{f(s_0)}\right| < e^M.$$

This inequality therefore holds inside the circle also. Hence the function

$$h(s) = \log\{g(s)/g(s_0)\},$$

where the logarithm is zero at $s = s_0$, and is regular for $|s - s_0| \leq \frac{1}{2}r$, and

$$h(s_0) = 0, \quad \mathbf{R}\{h(s)\} < M.$$

Hence, by Carathéodory's theorem,*

$$|h(s)| < AM \quad (|s - s_0| \leq \tfrac{3}{8}r),$$

and so, for $|s - s_0| \leq \frac{1}{4}r$,

$$|h'(s)| = \left|\frac{1}{2\pi i}\int_C \frac{h(z)}{(z-s)^2}\,dz\right| < \frac{AM}{r},$$

where C is the circle $|z - s| = \frac{1}{8}r$. This gives the result.

LEMMA. *If $f(s)$ satisfies the condition of the previous lemma, and has no zeros in the right half of the circle $|s - s_0| \leq r$, then*

$$(1) \qquad -\mathbf{R}\left\{\frac{f'(s_0)}{f(s_0)}\right\} < A_1\frac{M}{r};$$

while if $f(s)$ has a zero ρ_0 between $s_0 - \frac{1}{2}r$ and s_0, then

$$(2) \qquad -\mathbf{R}\left\{\frac{f'(s_0)}{f(s_0)}\right\} < A_1\frac{M}{r} - \frac{1}{s_0 - \rho_0}.$$

The previous lemma gives

$$-\mathbf{R}\left\{\frac{f'(s_0)}{f(s_0)}\right\} < A_1\frac{M}{r} - \Sigma\,\mathbf{R}\,\frac{1}{s_0-\rho},$$

and, since $\mathbf{R}\{1/(s_0 - \rho)\} \geq 0$ for every ρ, both results follow at once.

1.42. THEOREM 8. *There is a constant A_2 such that $\zeta(s)$ is not zero for*

$$\sigma \geq 1 - \frac{A_2}{\log t}, \quad t > t_0.$$

* See Landau's *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 1916, § 24.

This depends on an inequality of the same type as (14) of the Introduction. By (2″) of the Introduction

$$-\mathbf{R}\left\{\frac{\zeta'(s)}{\zeta(s)}\right\}=\sum_{p,m}\frac{\log p}{p^{m\sigma}}\cos(mt\log p).$$

Hence

$$(1)\qquad -3\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)}-4\mathbf{R}\frac{\zeta'(\sigma_0+i\gamma)}{\zeta(\sigma_0+i\gamma)}-\mathbf{R}\frac{\zeta'(\sigma_0+2i\gamma)}{\zeta(\sigma_0+2i\gamma)}$$

$$=\sum_{p,m}\frac{\log p}{p^{m\sigma}}\{3+4\cos(m\gamma\log p)+\cos(2m\gamma\log p)\}\geqq 0,$$

the expression in the bracket being never negative.

Let $\beta+i\gamma$ be a zero of $\zeta(s)$ for which, say, $\beta>\frac{3}{4}$. If there are no such zeros, the result, of course, follows at once.

Let $\sigma_0=1+a/\log\gamma$, where a is a constant to be determined later. We apply the last lemma to $f(s)=\zeta(s)$ and the circles with centres $\sigma_0+i\gamma$, $\sigma_0+2i\gamma$ and radius $\frac{1}{2}$. Neither of these circles has any zeros in its right half, and the former has at least one zero in its left half, if γ is large enough. Now

$$\left|\frac{1}{\zeta(s_0)}\right|\leqq\zeta(\sigma_0)<\frac{A}{\sigma_0-1}=A\log\gamma,$$

so that, by (9) of the Introduction,

$$\left|\frac{\zeta(s)}{\zeta(s_0)}\right|<At^{\frac{1}{2}}\log\gamma<A\gamma^{\frac{1}{2}}\log\gamma<e^{\log\gamma}\quad(\gamma>\gamma_0),$$

in $|s-s_0|\leqq\frac{1}{2}$. Hence, taking $r=\frac{1}{2}$, $M=\log\gamma$, in the lemma, we have

$$(2)\qquad -\mathbf{R}\left\{\frac{\zeta'(\sigma_0+2i\gamma)}{\zeta(\sigma_0+2i\gamma)}\right\}<2A_1\log\gamma,$$

$$(3)\qquad -\mathbf{R}\left\{\frac{\zeta'(\sigma_0+i\gamma)}{\zeta(\sigma_0+i\gamma)}\right\}<2A_1\log\gamma-\frac{1}{\sigma_0-\beta}.$$

Also, as $\sigma_0\to 1$,

$$-\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)}\sim\frac{1}{\sigma_0-1},$$

so that, for $\gamma>\gamma_1$,

$$(4)\qquad -\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)}<\frac{5}{4}\frac{1}{\sigma_0-1}.$$

We take $\frac{5}{4}$ as a convenient constant between 1 and the $\frac{4}{3}$ which emerges from (1). Hence (1) gives

$$\frac{15}{4}\frac{1}{\sigma_0-1}+8A_1\log\gamma-\frac{4}{\sigma_0-\beta}+2A_1\log\gamma>0.$$

Solving for $1-\beta$, we obtain

$$1-\beta > \left(\frac{4}{\frac{15}{4a}+10A_1} - a\right)\frac{1}{\log\gamma},$$

and the constant in brackets is positive if a is small enough. This proves the theorem.

1.43. THEOREM 9*. *There is a region*

$$1 - A/\log t \leqq \sigma \leqq 2, \quad t \geqq t_0,$$

in which we have uniformly

$$\frac{1}{\zeta(s)} = O(\log t), \quad \frac{\zeta'(s)}{\zeta(s)} = O(\log t).$$

In particular

$$\frac{1}{\zeta(1+it)} = O(\log t), \quad \frac{\zeta'(1+it)}{\zeta(1+it)} = O(\log t).$$

This corresponds to Theorem 6. Naturally it is more difficult to prove than Theorem 6, since it depends in the first place on the fact that $1/\zeta(s)$ is regular in the region considered. We shall deduce it from the following lemma†.

1.44. LEMMA. *Let $f(s)$ satisfy the conditions of the first lemma of* § 1.41, *and let*

(1) $$|f'(s_0)/f(s_0)| < M/r.$$

Suppose also that $f(s) \neq 0$ for

$$|s-s_0| \leqq r, \quad \sigma \geqq \sigma_0 - 2r',$$

where $0 < r' < \frac{1}{4}r$. Then

(2) $$|f'(s)/f(s)| < AM/r \quad (|s-s_0| \leqq r').$$

We take $s_0 = 0$. Then (1), and the lemma of § 1.41 with $s = 0$, give

(3) $$|\Sigma\, 1/\rho| < AM/r.$$

Now $$|\rho| \geqq -\mathbf{R}(\rho) > 2r' \geqq 2|s|, \text{ for } |s| \leqq r'.$$

Hence $$\left|\frac{1}{s-\rho}+\frac{1}{\rho}\right| = \frac{|s|}{|s-\rho|\,|\rho|} \leqq \frac{r'}{|\frac{1}{2}\rho|\,|\rho|} < \frac{-\mathbf{R}(\rho)}{|\rho|^2} = -\mathbf{R}\left(\frac{1}{\rho}\right),$$

and so $$\left|\Sigma\frac{1}{s-\rho} + \Sigma\frac{1}{\rho}\right| \leqq -\mathbf{R}\,\Sigma\left|\frac{1}{\rho}\right| \leqq \left|\Sigma\frac{1}{\rho}\right|,$$

so that, by (3), $$\left|\Sigma\frac{1}{s-\rho}\right| \leqq 2\left|\Sigma\frac{1}{\rho}\right| < \frac{AM}{r}.$$

The result now follows from the lemma of § 1.41.

* Gronwall (1), Landau (12). † Landau (12).

1.45. We can now prove Theorem 9. We know that $\zeta(s)$ is not zero in a region $\sigma \geqq 1 - A_2/\log t$, $t \geqq 2$. Let $s_0 = \sigma_0 + it_0$, where $\sigma_0 = 1 + A_2/(4 \log t_0)$. Then, for $t_0 > 3$, the circle $|s - s_0| \leqq \frac{1}{2}$ belongs to the region $\sigma > \frac{1}{2}$, $t > 2$, in which

$$\zeta(s) = O(\sqrt{t}) = O(\sqrt{t_0}).$$

Also

$$(1) \qquad \left|\frac{1}{\zeta(s_0)}\right| \leqq \Sigma \left|\frac{\mu(n)}{n^{s_0}}\right| \leqq \Sigma \frac{1}{n^{\sigma_0}} < \frac{A}{\sigma_0 - 1} < A \log t_0 .$$

Hence

$$(2) \qquad \left|\frac{\zeta(s)}{\zeta(s_0)}\right| < e^{A_3 \log t_0}.$$

Also

$$(3) \qquad \left|\frac{\zeta'(s_0)}{\zeta(s_0)}\right| \leqq \Sigma \frac{\Lambda(n)}{n^{\sigma_0}} = -\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} < \frac{A}{\sigma_0 - 1} < A_4 \log t_0 .$$

For $t_0 > A$, $\zeta(s)$ has no zeros in the part of the circle where

$$\sigma \geqq 1 - \tfrac{1}{2} A_2/\log t_0.$$

We can therefore apply the lemma, with

$$r = \tfrac{1}{2}, \quad r' = 3A_2/8 \log t_0, \quad M = \operatorname{Max}(A_3, 2A_4) \log t_0.$$

We deduce that

$$\frac{\zeta'(\sigma + it_0)}{\zeta(\sigma + it_0)} < A \log t_0 \qquad \left(1 - \frac{A_2}{8 \log t_0} \leqq \sigma \leqq 1 + \frac{A_2}{4 \log t_0}\right).$$

This is the desired result in the range stated; and for

$$1 + \tfrac{1}{4} A_2/\log t \leqq \sigma \leqq 2$$

the result follows at once from (3).

Again, for $\qquad \sigma \geqq 1 - A_2/8 \log t,$

$$\begin{aligned} \log \frac{1}{|\zeta(s)|} &= -\mathbf{R} \log \zeta(s) \\ &= -\mathbf{R} \log \zeta\left(1 + \frac{A_2}{4 \log t} + it\right) + \int_\sigma^{1 + A_2/4 \log t} \mathbf{R} \frac{\zeta'(u + it)}{\zeta(u + it)} dt \\ &\leqq \log \zeta\left(1 + \frac{A_2}{4 \log t}\right) + \frac{3A_2}{8 \log t} A \log t \\ &< \log \log t + A. \end{aligned}$$

Hence $\qquad \dfrac{1}{\zeta(s)} = O(\log t) \qquad \left(1 - \dfrac{A_2}{8 \log t} \leqq \sigma \leqq 1 + \dfrac{A_2}{4 \log t}\right),$

while for $1 + \frac{1}{4} A_2/\log t \leqq \sigma \leqq 2$ the result follows at once from (1).

1.46. THEOREM 10. $\dfrac{1}{\zeta(1+it)} = \Omega(\log\log t)$.

The deduction of this from Theorem 5 is the same as that of Theorem 7 from Theorem 2.

1.5. *$\zeta(s)$ in the critical strip.*

1.51. The functional equation obtained in the Introduction enables us to deduce the results for the half-plane $\sigma \leqq 0$ from those we have obtained in the half-plane $\sigma \geqq 1$. It gives, for σ fixed and $t > 0$,

$$|\zeta(s)| = 2^{\sigma}\pi^{\sigma-1}\{\tfrac{1}{2}e^{\frac{1}{2}\pi t} + O(e^{-\pi t})\}\, t^{\frac{1}{2}-\sigma} e^{-\frac{1}{2}\pi t + O(1)}\, |\zeta(1-s)|$$

$$= t^{\frac{1}{2}-\sigma} e^{O(1)} |\zeta(1-s)|. \tag{1}$$

For example, we deduce from Theorem 6 that $\zeta(it) = O(t^{\frac{1}{2}}\log t)$.

The remaining results may be left to the reader.

It is plain that we can obtain no result of the kind considered for $1/\zeta(s)$ in the critical strip $0 < \sigma < 1$. For, on any line $\sigma = \sigma_0\ (0 < \sigma_0 < 1)$, $1/\zeta(s)$ may, for all we know, become infinite an infinity of times. We therefore confine our attention to $\zeta(s)$.

1.52. We now introduce a function $\mu(\sigma)$ which gives the order of $\zeta(\sigma + it)$ as a function of t; more precisely, $\mu(\sigma)$ is the lower bound of numbers λ such that $\zeta(\sigma + it) = O(t^{\lambda})$.

It follows from Theorem 1 that $\mu(\sigma) = 0$ for $\sigma > 1$, and then from the functional equation that $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma < 0$. Further, it follows from (9) of the Introduction that $\mu(\sigma)$ is finite for $0 \leqq \sigma \leqq 1$.

The most important property of $\mu(\sigma)$ is that it is convex downwards. This is a direct consequence of Lindelöf's theorem, and is discussed in detail by Hardy and Riesz. It tells us a good deal about $\mu(\sigma)$ in $0 < \sigma < 1$, but does not enable us to determine it completely.

In the first place, we must have $\mu(\sigma) \geqq 0$ for $\frac{1}{2} \leqq \sigma < 1$, or the function would cease to be convex somewhere between $\frac{1}{2}$ and 1. Similarly $\mu(\sigma) \geqq \frac{1}{2} - \sigma$ for $0 < \sigma < \frac{1}{2}$.

Again, $\mu(\sigma) \leqq \frac{1}{2} - \frac{1}{2}\sigma$ for $0 < \sigma < 1$; for the property of convexity shews that $\mu(\sigma)$ does not rise above the straight line joining the point $(0, \frac{1}{2})$, which gives its value for $\sigma = 0$, to $(1, 0)$, which gives its value for $\sigma = 1$.

We shall see later that we can obtain a slightly smaller upper bound for $\mu(\sigma)$; but its exact value is not known anywhere in the critical strip.

1.53. The above argument shews that $\zeta(s) = O(t^{\frac{1}{2}-\frac{1}{2}\sigma+\epsilon})$ in the critical strip. We can however obtain a slightly more precise result.

THEOREM 11. $$\zeta(s) = O(t^{\frac{1}{2}-\frac{1}{2}\sigma}\log t)$$
uniformly in $0 \leqq \sigma \leqq 1$. *In particular*
$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{2}}\log t).$$

We use the more general form* of Lindelöf's theorem, which states that, *if* $f(s)$ *is regular and of the form* $O(t^A)$ *for* $\sigma_1 \leqq \sigma \leqq \sigma_2$, *and*
$$f(\sigma_1+it) = O(t^{k_1}),\quad f(\sigma_2+it) = O(t^{k_2}),$$
then $f(s) = O\{t^{k(\sigma)}\}$ *uniformly for* $\sigma_1 \leqq \sigma \leqq \sigma_2$, $k(\sigma)$ *being the linear function of* σ *which assumes the values* k_1, k_2, *for* $\sigma = \sigma_1, \sigma_2$.

Now $f(s) = \zeta(s)/\log s$ satisfies the conditions (for $t > 1$) with $\sigma_1 = 0$, $\sigma_2 = 1$, $k_1 = \frac{1}{2}$, $k_2 = 0$. The result follows at once from this.

1.6. *Weyl's method of approximation and the theorems of Hardy and Littlewood*†.

1.61. We have already remarked that it is possible to improve on the O-result of the foregoing work, and we shall now give an account of the method used. The analysis is very complicated, and we have only space to give a brief outline of it here.

The analysis is based on some very important inequalities, due to Weyl, who obtained inequalities satisfied by sums of the form
$$S = \sum_{r=0}^{\mu-1} e^{iP(r)},$$
where μ is large, and $P(r)$ is a polynomial in r with real coefficients. If we replace each term in S by its modulus, we obtain $|S| \leqq \mu$. But it is easily seen that, in simple cases, more than this is true. Thus if $P(r) = ar$, then
$$|S| = |(1-e^{ia\mu})/(1-e^{ia})| \leqq |\operatorname{cosec}\tfrac{1}{2}a|,$$
and so $$|S| \leqq \operatorname{Min}(\mu, |\operatorname{cosec}\tfrac{1}{2}a|).$$

If $P(r) = ar^2$, then
$$|S|^2 = S\bar{S} = \sum_{q=0}^{\mu-1}\sum_{r=0}^{\mu-1} e^{ia(q^2-r^2)} = \sum_{n=-\mu+1}^{\mu-1}\sum_{r} e^{ian(2r+n)},$$
where $n = q - r$, and in the last sum r runs through $\mu - |n|$ consecutive integers. Hence
$$|S|^2 \leqq \sum_{n=-\mu+1}^{\mu-1}\left|\sum_{r} e^{2ianr}\right| \leqq \sum_{n=-\mu+1}^{\mu-1} \operatorname{Min}(\mu, |\operatorname{cosec} na|).$$

We can continue this process, and obtain a similar inequality in the

* Hardy and Riesz, *loc. cit.*

† Weyl (1), Littlewood (2), Landau (10).

general case. Actually our object is to obtain an inequality for the sum

$$S' = \sum_{n=N}^{N'} n^{it}$$

which is clearly connected with the zeta-function. To do this we divide S' up into partial sums, each containing μ terms. The modulus of the first of these, for example, is

$$\left|\sum_{n=N}^{N+\mu-1} n^{it}\right| = \left|\sum_{r=0}^{\mu-1}\left(1+\frac{r}{N}\right)^{it}\right|.$$

Now

$$\left(1+\frac{r}{N}\right)^{it} = \exp\left\{it\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{r^n}{N^n}\right\} = \exp\left\{it\sum_{n=1}^{k}\right\}\exp\left\{it\sum_{n=k+1}^{\infty}\right\}$$

$$= e^{iP(r)}\sum_{\nu=0}^{\infty} e_\nu(t)\frac{r^\nu}{N^\nu},$$

where $P(r)$ is a polynomial of degree k, and $e_\nu(t)$ is a function of ν, k, and t with which we can easily deal. Hence

$$\sum_{r=0}^{\mu-1}\left(1+\frac{r}{N}\right)^{it} = \sum_{\nu=0}^{\infty} e_\nu(t)\,N^{-\nu}\sum_{r=0}^{\mu-1} r^\nu e^{iP(r)}.$$

Here the inner sum is, apart from the factor r^ν, of the form S, and we can obtain an inequality for it from that for S by partial summation. This establishes the connection between S and S'.

The actual result* is as follows. Let N and N' be positive integers such that $N \leqq N' \leqq 2N$; let $t > 3$, and $K = 2^{k-1}$. Then

$$(1)\quad \left|\sum_{n=N}^{N'} n^{it}\right| < A\,\{N^{1-\frac{1}{K}}t^{\frac{1}{(k+1)K}} + Nt^{-\frac{1}{(k+1)K}}\log^{\frac{k-1}{K}} N\}\log^{\frac{1}{K}} t.$$

The inequality contains the arbitrary integer k, which may be chosen to suit any particular case under consideration.

We deduce from (1) the following inequalities†:

$$(2)\quad \left|\sum_{t^{\frac{2}{k+2}} < n \leqq t^{\frac{2}{k+1}}} n^{-s}\right| < At^{-\frac{1}{8kK}}\log^2 t \quad (k \geqq 3,\ 1-1/(4K) \leqq \sigma \leqq 1),$$

$$(3)\quad \left|\sum_{\sqrt{t} < n \leqq t} n^{-s}\right| < A \qquad (\tfrac{31}{32} \leqq \sigma < 1),$$

$$(4)\quad \left|\sum_{t^{2/r} < n \leqq t} n^{-s}\right| < A \quad \left(6 \leqq r \leqq \log\log t,\ R = 2^{r-1},\ 1-\frac{1}{R} \leqq \sigma < 1\right).$$

These results are obtained from (1) by dividing the sums into parts $N \leqq n \leqq N'$, in which, as in (1), $N' \leqq 2N$, and using partial summation.

* Landau (10), p. 110, (16).

† See Landau 10, pp. 114–115. We state (3) and (4) with upper limit t instead of t^2, this being all that is required in the method followed here.

1.62. We can now prove*

THEOREM 12. *We have*

$$(1) \qquad \zeta(s) = O\{t^{4(1-\sigma)/\{\log 1/(1-\sigma)\}} \log t/\log\log t\}$$

uniformly for $63/64 \leqq \sigma < 1$; *and*†

$$(2) \qquad \zeta(1+it) = O(\log t/\log\log t).$$

Let r be the integral part of

$$\operatorname{Min}\,[\log\{1/(1-\sigma)\}/\log 2,\ \log\log t],$$

so that $6 \leqq r \leqq \log\log t$ for $t > A$. Since

$$\sigma \geqq 1 - 2^{-r} > 1 - 1/R,$$

we can apply 1.61 (4), and obtain from Theorem 19, with $x = t$,

$$\zeta(s) = O(1) + O\left(\sum_{n \leqq t^{2/r}} n^{-s}\right) = O(1) + O\left(\sum_{n \leqq t^{2/r}} n^{-\sigma}\right).$$

Suppose first that

$$1 - \sigma \leqq \log\log t/\log t.$$

Then it is easily seen that, for $t > A$,

$$r \geqq \tfrac{1}{2}\log\log t.$$

Hence
$$\sum_{n \leqq t^{2/r}} n^{-\sigma} = \sum_{n \leqq t^{2/r}} n^{1-\sigma} n^{-1} \leqq t^{2(1-\sigma)/r} \sum_{n \leqq t^{2/r}} n^{-1}$$

$$< \exp\left\{\log t \frac{4}{\log\log t}\cdot\frac{\log\log t}{\log t}\right\} \log t^{2/r} = O\left(\frac{\log t}{r}\right) = O\left(\frac{\log t}{\log\log t}\right),$$

which is of the required form.

If, on the other hand,

$$1 - \sigma > \log\log t/\log t,$$

then
$$r \geqq \left[\log\frac{1}{1-\sigma}\right] > \tfrac{1}{2}\log\frac{1}{1-\sigma},$$

for $t > A$; and

$$\sum_{n \leqq t^{2/r}} n^{-\sigma} < \int_0^{t^{2/r}} u^{-\sigma}\,du = \frac{t^{2(1-\sigma)/r}}{1-\sigma},$$

which is of the form of the right-hand side of (1). This proves (1), and (2) follows by continuity.

* It is convenient at this point to anticipate the result of Theorem 19. It does not seem appropriate to set it out in full in the middle of a sketch of this kind, and in any case the results could be obtained without it. A similar remark applies a little later to the use of Theorem 22.

† Hardy and Littlewood (1), Weyl (1).

1.63. THEOREM 13. *There is a constant A_5 such that $\zeta(s) \neq 0$ for*

$$\sigma \geqq 1 - A_5 \log\log t/\log t, \quad t \geqq t_0.$$

The previous theorem gives, for

$$\sigma \geqq 1 - (\log\log t)^2/\log t, \quad t > A,$$

$$(1)\quad \zeta(s) = O\left[\exp\left\{4\log t\,\frac{(\log\log t)^2}{\log t}\Big/\log\left(\frac{\log t}{(\log\log t)^2}\right)\right\}\frac{\log t}{\log\log t}\right]$$

$$= O\{\exp(A\log\log t)\log t/\log\log t\} = O(\log^A t).$$

The proof now depends on the same ideas as the proof of Theorem 8, but we use the additional fact that (1) holds for

$$\sigma \geqq 1 - (\log\log t)^2/\log t,$$

and not merely (as we knew before) for $\sigma \geqq 1 - 1/\log t$. Let

$$\sigma_0 = 1 + a\log\log\gamma/\log\gamma \quad (\gamma > 3).$$

It is sufficient to consider those zeros $\beta + i\gamma$, if there are any, for which

$$\beta > \sigma_0 - \tfrac{1}{2}(\log\log\gamma)^2/\log\gamma, \quad \gamma > A;$$

for any other zeros certainly satisfy the required conditions.

For $s_0 = \sigma_0 + i\gamma$, and also for $s_0 = \sigma_0 + 2i\gamma$, the circle

$$|s - s_0| \leqq r = (\log\log\gamma)^2/\log\gamma$$

lies in the region $\sigma \geqq 1 - (\log\log t)^2/\log t, \quad t \geqq t_0,$

if γ is sufficiently large. Now

$$\left|\frac{1}{\zeta(s_0)}\right| < \frac{A}{\sigma_0 - 1} = \frac{A\log\gamma}{\log\log\gamma},$$

and it follows from this and (1) that, for

$$|s - s_0| \leqq r, \quad \gamma > A,$$

$$(2)\qquad \left|\frac{\zeta(s)}{\zeta(s_0)}\right| < (\log\gamma)^A.$$

Hence, by the second lemma of § 1.41,

$$-\mathbf{R}\left\{\frac{\zeta'(\sigma_0 + i\gamma)}{\zeta(\sigma_0 + i\gamma)}\right\} < \frac{A_1A_6\log\log\gamma}{(\log\log\gamma)^2/\log\gamma} - \frac{1}{\sigma_0 - \beta},$$

$$-\mathbf{R}\left\{\frac{\zeta'(\sigma_0 + 2i\gamma)}{\zeta(\sigma_0 + 2i\gamma)}\right\} < \frac{A_1A_6\log\log\gamma}{(\log\log\gamma)^2/\log\gamma}.$$

Also 1.42 (4) holds as before, and we deduce from 1.42 (1) that

$$\frac{15}{4}\frac{1}{\sigma_0 - 1} + \frac{5A_1A_6\log\gamma}{\log\log\gamma} - \frac{4}{\sigma_0 - \beta} > 0.$$

Solving for $1 - \beta$ we obtain

$$(1 - \beta)\frac{\log\gamma}{\log\log\gamma} > \frac{4}{\dfrac{15}{4a} + 5A_1A_6} - a,$$

which is positive if a is sufficiently small. This proves the theorem.

1.64. Apart from the extension of 1.63 (1) to the wider range, the analytical machinery of the proof is the same as in the proof that $\zeta(s) \neq 0$ for $\sigma \geqq 1 - A/\log t$. It seems clear that until we discover some new analytical method, we cannot hope to extend the result very far—at most not beyond the immediate neighbourhood of the line $\sigma = 1$. For suppose that we use the above argument with any values of r and σ_0, and that

$$|\zeta(s)/\zeta(s_0)| < M$$

in the appropriate regions. We obtain

$$\frac{15}{4}\frac{1}{\sigma_0 - 1} + \frac{5A_1 \log M}{r} - \frac{4}{\sigma_0 - \beta} > 0,$$

or
$$1 - \beta > (\sigma_0 - 1)\left\{\frac{4}{\frac{15}{4} + \{5A_1(\sigma_0 - 1)\log M\}/r} - 1\right\}.$$

For the expression on the right to be positive, we must have

$$5A_1(\sigma_0 - 1)\log M < \tfrac{1}{4}r < \tfrac{1}{8},$$

and so, since $\log M$ is certainly large for large γ, $\sigma_0 - 1$ must be small; but then the lower limit obtained for $1 - \beta$ is small, i.e. our result is restricted to the immediate neighbourhood of $\sigma = 1$.

1.65. THEOREM 14. *There is a region*

$$1 - A\log\log t/\log t \leqq \sigma \leqq 2, \quad t > t_0,$$

in which we have uniformly

$$\frac{1}{\zeta(s)} = O\left(\frac{\log t}{\log\log t}\right), \quad \frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\log t}{\log\log t}\right).$$

In particular

$$\frac{1}{\zeta(1+it)} = O\left(\frac{\log t}{\log\log t}\right), \quad \frac{\zeta'(1+it)}{\zeta(1+it)} = O\left(\frac{\log t}{\log\log t}\right).$$

The proof is similar to that of Theorem 9. Let

$$s_0 = 1 + \frac{A_5 \log\log t_0}{4\log t_0} + it_0, \quad r = \frac{(\log\log t_0)^2}{2\log t_0},$$

where A_5 is the constant of Theorem 13. Then, by 1.63 (1), for $|s - s_0| \leqq r$,

$$\left|\frac{\zeta(s)}{\zeta(s_0)}\right| < \frac{\log^A t_0}{\sigma_0 - 1} < e^{A_7 \log\log t_0},$$

$$\left|\frac{\zeta'(s_0)}{\zeta(s_0)}\right| < \frac{A}{\sigma_0 - 1} = A\frac{\log t_0}{\log\log t_0} = \frac{A_8 \log\log t_0}{r}.$$

Also the part of the circle where

$$\sigma \geqq 1 - \tfrac{1}{2}A_5 \log\log t_0/\log t_0$$

belongs to the region where $\zeta(s) \neq 0$. We may therefore apply the lemma of § 1.44 with the above values of s_0 and r, and

$$r' = \tfrac{3}{8} A_5 \log\log t_0/\log t_0, \quad M = \operatorname{Max}(A_7, A_8) \log\log t_0.$$

All the results now follow exactly as in the proof of Theorem 9.

1.66. As a final application of the method, we extend our O-result for $\zeta(s)$ to values of σ not covered by Theorem 12, and in fact obtain a more precise result for certain values of σ, but not one which holds uniformly as $\sigma \to 1$.

THEOREM 15. *For fixed* $q \geqq 2$, $Q = 2^{q-1}$, $\sigma = 1 - 1/Q$,

$$(1) \qquad \zeta(s) = O\left(t^{\frac{1}{(q+1)Q}} \log^{1+\frac{1}{Q}} t\right).$$

In particular

$$(2) \qquad \zeta(\tfrac{1}{2} + it) = O(t^{\frac{1}{6}} \log^{\frac{3}{2}} t).$$

We deduce from our fundamental inequalities 1.61 (1) two further inequalities*:

$$(3) \qquad \sum_{n \leqq t^{\frac{2}{q+1}} / \log^{q-1} t} n^{-s} = O\left(t^{\frac{1}{(q+1)Q}} \log^{1+\frac{1}{Q}} t\right) \quad (q \geqq 2, \sigma = 1 - 1/Q),$$

$$(4) \qquad \sum_{t^{\frac{2}{q+1}} / \log^{q-1} t < n \leqq t} n^{-s} = O\left(t^{\frac{1}{(q+1)Q}}\right) \quad (q > 2, \sigma = 1 - 1/Q).$$

Suppose first that $q > 2$. Then, by Theorem 19,

$$\zeta(s) = \sum_{n<t} n^{-s} + O(1),$$

and dividing the sum into the parts (3) and (4), we obtain the result.

The case $q = 2$ is more difficult, and we have to appeal to some such result as Theorem 22. Theorem 22 gives, for example,

$$\zeta(\tfrac{1}{2} + it) = \sum_{n < \sqrt{(t/2\pi)}} n^{-\frac{1}{2} - it} + \chi \sum_{n < \sqrt{(t/2\pi)}} n^{-\frac{1}{2} + it} + O(1),$$

where $\chi = O(1)$; and the result follows on applying (3) to each of these sums.

1.67. Theorem 15 gives us a number of points below which the graph of the function $y = \mu(\sigma)$ of § 1.52 must pass. We now see that $\mu(\tfrac{1}{2}) \leqq \tfrac{1}{6}$, and that the graph touches the line $y = 0$ at $\sigma = 1$.

The actual value of $\mu(\sigma)$ in the critical strip remains unknown. But we shall see later that there is some reason to suppose that $\mu(\sigma) = 0$ for $\tfrac{1}{2} \leqq \sigma \leqq 1$, with a corresponding value (deduced from the functional equation) for $0 \leqq \sigma \leqq \tfrac{1}{2}$.

* See Landau (10), p. 118.

1.7. *Van der Corput's method**.

The inequality 1.66 (2) for $\zeta(\frac{1}{2}+it)$, deep as it is, is not the best that is known. A method similar in principle to that of Weyl, but still more complicated, has been invented by van der Corput, primarily for application to the divisor problem and similar problems in the theory of numbers. It has been shewn by Walfisz that van der Corput's method gives the following result:

THEOREM 16. $$\zeta(\tfrac{1}{2}+it)=O(t^{\frac{163}{988}}).$$

Still further applications of van der Corput's method have also been announced†.

1.8. Ω-*results for* $\zeta(s)$ *in the critical strip.*

1.81. It is easily seen that $\zeta(\sigma+it)$ is unbounded for every σ in $\frac{1}{2}\leqq\sigma<1$; in fact the proof that $\zeta(1+it)$ is unbounded applies equally well to the general case. But we can prove more than this.

THEOREM 17‡. *For* $\frac{1}{2}\leqq\sigma<1$,
$$\zeta(\sigma+it)=\Omega\{\exp(\log t)^{1-\sigma-\epsilon}\}.$$

Thus $|\zeta(\sigma+it)|$ takes values greater than any power of $\log t$, however large; but we cannot prove that it takes values as great as a power of t, however small.

We shall not give the proof, which is somewhat complicated; but we shall refer to it again at the end of Chapter II, where it is really more in place.

1.82. The above theorem tells us that in the critical strip $\zeta(s)$ is sometimes large. But it tells us nothing about the distribution of the values of t for which it is large. There is however another result which states a much weaker inequality, but states it for many more values of t. It may be expressed roughly by saying that $|\zeta(s)|$ is, in general, not very small.

THEOREM 18 §. *If H is any number greater than unity, then*
$$|\zeta(s)|>T^{-AH}$$
for $$-1\leqq\sigma\leqq 2,\quad T-\tfrac{1}{2}\leqq t\leqq T+\tfrac{1}{2},$$
except possibly for a set of values of t of measure $1/H$.

* See Walfisz (1).

† See a footnote on page 400 of *Math. Zeitschrift*, 29 (1928).

‡ Titchmarsh (4). § Valiron (1), Landau (5) and (14), Hoheisel (1).

1.83. We first prove the following lemma, which is often useful.

LEMMA. *If* $\rho = \beta + i\gamma$ *runs through zeros of* $\zeta(s)$,

$$\frac{\zeta'(s)}{\zeta(s)} - \sum_{|t-\gamma|<1} \frac{1}{s-\rho} = O(\log t) \tag{1}$$

uniformly for $-1 \leqq \sigma \leqq 2$.

Take $s_0 = \frac{3}{2} + it$, $r = 10$ in the first lemma of § 1.41. Then $M = A \log t$, and we obtain

$$\frac{\zeta'(s)}{\zeta(s)} - \sum_{|\rho - s_0| \leqq 5} \frac{1}{s-\rho} = O(\log t) \tag{2}$$

for $|s - s_0| \leqq \frac{5}{2}$, and in particular for $-1 \leqq \sigma \leqq 2$.

Finally, by (19) of the Introduction, the number of terms in one of the above sums but not in the other is $O(\log t)$, and each such term is $O(1)$.

1.84. We have, for

$$-1 \leqq \sigma \leqq 2, \quad T - \tfrac{1}{2} \leqq t \leqq T + \tfrac{1}{2},$$

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|T-\gamma|<1} \frac{1}{s-\rho} + O(\log T),$$

since the number of terms included in this sum, but not in that of the lemma, or vice versa, is $O(\log T)$, and each such term is $O(1)$. Hence

$$\log|\zeta(s)| = -\mathbf{R}\int_s^{2+it} \frac{\zeta'(z)}{\zeta(z)}\,dz + \log|\zeta(2+it)|$$

$$= \sum_{|T-\gamma|<1} \log|s-\rho| + O(\log T)$$

$$> \sum_{|T-\gamma|<1} \log|t-\gamma| + O(\log T). \tag{1}$$

It is easily seen from a graph that the integral

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \log|t-\gamma|\,dt,$$

considered as a function of γ, is a minimum when $\gamma = T$; and it is then equal to $-\log 2 - 1$. Since there are $O(\log T)$ terms in the sum, it follows that

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \sum_{|T-\gamma|<1} \log|t-\gamma|\,dt > -A\log T.$$

Hence

$$\sum_{|T-\gamma|<1} \log|t-\gamma| > -AH\log T, \tag{2}$$

except in a set of measure $1/H$, and the result follows from (1) and (2).

The exceptional values of t are, of course, those in the neighbourhood of ordinates of zeros of $\zeta(s)$.

1.85. *Application to a formula of Ramanujan**.

Let α and β be real numbers such that $\alpha\beta = \pi$, and consider the integral

$$\frac{1}{2\pi i}\int \alpha^{-2s}\frac{\Gamma(s)}{\zeta(1-2s)}\,ds = \frac{1}{2\pi i}\int \frac{\beta^{2s}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds$$

taken round the rectangle $1 \pm iT$, $-\frac{1}{2} \pm iT$ (the two forms are equivalent on account of the functional equation). To obtain a suitable set of values through which we can make $T \to \infty$, it is most convenient to use, not Theorem 18 itself, but 1.84 (1). Suppose that $t \to \infty$ through values such that $|t-\gamma| > \exp(-A_9\gamma/\log\gamma)$ for every ordinate γ of a zero of $\zeta(s)$. Then for $t < t_0$

$$\log|\zeta(s)| > -\sum_{|t-\gamma|<1} A_9\gamma/\log\gamma + O(\log t) > -A_{10}t,$$

where $A_{10} < \frac{1}{4}\pi$ if A_9 is small enough, and t_0 large enough.

It now follows from the asymptotic formula for the Γ-function that the integral along the horizontal side of the contour tends to zero as $T \to \infty$ through values such that $|T-\gamma| > \exp(-A_9\gamma/\log\gamma)$ for every γ. Hence by the theorem of residues†,

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \alpha^{-2s}\frac{\Gamma(s)}{\zeta(1-2s)}\,ds - \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{\beta^{2s}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds$$
$$= -\frac{1}{2\sqrt{\pi}}\sum \beta^{\rho}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}.$$

The first term on the left is equal to

$$\sum\frac{\mu(n)}{n}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\left(\frac{n}{\alpha}\right)^{2s}\Gamma(s)\,ds = -\sum\frac{\mu(n)}{n}\{1-e^{-(\alpha/n)^2}\} = \sum\frac{\mu(n)}{n}e^{-(\alpha/n)^2},$$

and, evaluating the second term in the same way, and multiplying through by $\sqrt{\alpha}$, we obtain Ramanujan's result

$$\sqrt{\alpha}\sum\frac{\mu(n)}{n}e^{-(\alpha/n)^2} - \sqrt{\beta}\sum\frac{\mu(n)}{n}e^{-(\beta/n)^2} = -\frac{1}{2\sqrt{\beta}}\sum\beta^{\rho}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}.$$

We have, of course, not proved that the series on the right is convergent in the ordinary sense. We have merely proved that it is convergent if the terms are bracketed in such a way that two terms for which

$$|\gamma-\gamma'| < \exp(-A_9\gamma/\log\gamma) + \exp(-A_9\gamma'/\log\gamma')$$

are included in the same bracket. Of course the zeros are, on the average, much farther apart than this, and it is quite possible that the series may converge without any bracketing. But we are unable to prove this, even on the Riemann hypothesis.

* Hardy and Littlewood (2), pp. 156–159.

† In forming the series of residues we have supposed for simplicity that the poles are all simple.

CHAPTER II

MEAN VALUE THEOREMS

2.11. The problem of the order of $\zeta(s)$ in the critical strip is, as we have seen, unsolved. The problem of the average order, or mean value, is much easier, and, in its simplest form, has been solved completely. The form which it takes is that of determining the behaviour of

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2 dt$$

as $T \to \infty$, for any given value of σ. We also consider mean values of other powers of $|\zeta(s)|$. The results give an interesting comparison with those of Chapter I, and indeed might be used to prove O-results if we could push them far enough. Apart from this they have applications to the problem of the zeros, and also to problems in the theory of numbers*.

For $\sigma > 1$ we deduce at once, from the general mean value theorem of § 1.23, that

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2 dt = \Sigma n^{-2\sigma} = \zeta(2\sigma).$$

We shall shew that this is also true for $\frac{1}{2} < \sigma \leqq 1$. But we first require an approximate formula for $\zeta(s)$ in the critical strip.

2.12. THEOREM 19†. *We have*

$$(1) \qquad \zeta(s) = \sum_{n \leqq x} n^{-s} - x^{1-s}/(1-s) + O(x^{-\sigma}),$$

uniformly for $\sigma \geqq \sigma_0 > 0$, $|t| < 2\pi x/c$, *where* $c > 1$.

We may suppose without loss of generality that x is half an odd integer, since the last term in the sum (which might be affected by the restriction) is $O(x^{-\sigma})$, and so is the possible variation in $x^{1-s}/(1-s)$.

Suppose first that $\sigma > 1$. Then a simple application of the theorem of residues shews that

$$\zeta(s) - \sum_{n<x} n^{-s} = \sum_{n>x} n^{-s} = -\frac{1}{2i}\int_{x-i\infty}^{x+i\infty} z^{-s}\cot\pi z\, dz$$

$$(2) \quad = -\frac{1}{2i}\int_{x-i\infty}^{x} (\cot\pi z - i) z^{-s} dz - \frac{1}{2i}\int_{x}^{x+i\infty} (\cot\pi z + i) z^{-s} dz - \frac{x^{1-s}}{1-s}.$$

* Hardy and Littlewood (4).

† Hardy and Littlewood (3) and (4). In (3) quite a different proof, by a "real variable" method, is given.

The final formula holds, by the theory of analytic continuation, for all values of s, since the last two integrals are uniformly convergent in any finite region. In the second integral we put $z = x + ir$, so that

$$|\cot \pi z + i| = 2/(1 + e^{2\pi r}) < 2e^{-2\pi r},$$
$$|z^{-s}| = |z|^{-\sigma} e^{t \operatorname{am} z} < x^{-\sigma} e^{|t| \arctan r/x} < x^{-\sigma} e^{|t| r/x}.$$

Hence the modulus of this term does not exceed

$$x^{-\sigma} \int_0^\infty e^{-2\pi r + |t| r/x} \, dr = \frac{x^{-\sigma}}{2\pi - |t|/x}.$$

A similar result holds for the other integral, and the theorem follows.

2.13. We also require the following lemma.

LEMMA. *We have*

$$\sum_{0<m<n<T} \sum \frac{1}{m^\sigma n^\sigma \log n/m} = O(T^{2-2\sigma} \log T)$$

for $\frac{1}{2} \leqq \sigma < 1$, *and uniformly for* $\frac{1}{2} \leqq \sigma \leqq \sigma_0 < 1$.

Let Σ_1 denote the sum of the terms for which $m < \frac{1}{2}n$, Σ_2 the remainder. In Σ_1, $\log n/m > A$, so that

$$\Sigma_1 < A \sum_{m<n<T} \sum m^{-\sigma} n^{-\sigma} < A (\sum_{n<T} n^{-\sigma})^2 < A T^{2-2\sigma}.$$

In Σ_2 we write $m = n - r$, where $1 \leqq r \leqq \frac{1}{2}n$, and then

$$\log n/m = -\log(1 - r/n) > r/n.$$

Hence

$$\Sigma_2 < A \sum_{n=1}^{T} \sum_{r=1}^{\frac{1}{2}n} (n-r)^{-\sigma} n^{-\sigma} n/r < A \sum_{n=1}^{T} n^{1-2\sigma} \sum_{r=1}^{\frac{1}{2}n} r^{-1} < A T^{2-2\sigma} \log T.$$

2.14. We can now prove

THEOREM 20. *As* $T \to \infty$,

$$\int_1^T |\zeta(\sigma + it)|^2 \, dt \sim T\zeta(2\sigma) \qquad (\sigma > \tfrac{1}{2}).$$

We have already accounted for the case $\sigma > 1$, so that we now suppose that $\frac{1}{2} < \sigma \leqq 1$. Since $t > 1$, Theorem 19, with $x = t$, gives

$$\zeta(s) = \sum_{n<t} n^{-s} + O(t^{-\sigma}) = Z + O(t^{-\sigma}),$$

say. Now

$$\begin{aligned}
\int_1^T |Z|^2 dt &= \int_1^T \{ \sum_{m<t} m^{-\sigma - it} \sum_{n<t} n^{-\sigma+it} \} \, dt \\
&= \sum_{m<T} \sum_{n<T} m^{-\sigma} n^{-\sigma} \int_{T_1}^T (n/m)^{it} \, dt \qquad (T_1 = \operatorname{Max}(m, n)) \\
&= \sum_{n<T} n^{-2\sigma} (T - n) + O\{ \sum_{m<n<T} \sum m^{-\sigma} n^{-\sigma} / \log n/m \} \\
&= T \sum_{n<T} n^{-2\sigma} + O(T^{2-2\sigma}) + O(T^{2-2\sigma} \log T) \\
&\sim T\zeta(2\sigma),
\end{aligned}$$

provided that $\sigma < 1$. If $\sigma = 1$, we can replace σ, in the O-term, by $\frac{3}{4}$, say, and obtain the same result. Hence, in any case,

$$\int_1^T |\zeta(s)|^2 dt = \int_1^T |Z|^2 dt + O\left\{\int_1^T |Z|\, t^{-\sigma} dt\right\} + O\left\{\int_1^T t^{-2\sigma} dt\right\}$$

$$= \int_1^T |Z|^2 dt + O\left\{\int_1^T |Z|^2 dt \int_1^T t^{-2\sigma} dt\right\}^{\frac{1}{2}} + O(\log T)$$

$$= \int_1^T |Z|^2 dt + O\{(T\log T)^{\frac{1}{2}}\} + O(\log T),$$

and the result follows.

2.15. It will be useful later to have a result of this type which holds uniformly in the strip. It is

THEOREM 21*

$$\int_1^T |\zeta(\sigma + it)|^2 dt < A T \operatorname{Min}\left\{\log T, \frac{1}{\sigma - \frac{1}{2}}\right\}$$

uniformly for $\frac{1}{2} \leqq \sigma \leqq 2$.

Suppose first that $\frac{1}{2} \leqq \sigma \leqq \frac{3}{4}$. Then we have, as before,

$$\int_1^T |Z|^2 dt < T \sum_{n<T} n^{-2\sigma} + O(T^{2-2\sigma} \log T)$$

uniformly in σ. Now

$$\sum_{n<T} n^{-2\sigma} \leqq \sum_{n<T} n^{-1} < A \log T,$$

and also
$$\sum_{n<T} n^{-2\sigma} < 1 + \int_1^\infty u^{-2\sigma} du < \frac{A}{\sigma - \frac{1}{2}}.$$

Similarly
$$T^{2-2\sigma} \log T \leqq T \log T,$$

and also, putting
$$x = (2\sigma - 1) \log T,$$

$$T^{2-2\sigma} \log T = \tfrac{1}{2} T x e^{-x}/(\sigma - \tfrac{1}{2}) \leqq \tfrac{1}{2} T/(\sigma - \tfrac{1}{2}).$$

This gives the result for $\sigma \leqq \frac{3}{4}$, the term $O(t^{-\sigma})$ being dealt with as before.

If $\frac{3}{4} \leqq \sigma \leqq 2$, we obtain

$$\int_1^T |Z|^2 dt < T \sum n^{-\frac{3}{2}} + O(T^{\frac{1}{2}} \log T),$$

whence the result.

2.16. The particular case $\sigma = \frac{1}{2}$ is

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^2 dt = O(T \log T).$$

* Littlewood (4).

We can improve the O-result to an asymptotic equality. But Theorem 19 is not sufficient for this purpose, and we require a deeper result of the same kind.

2.21. *The approximate functional equation**.

The weakness of Theorem 19 lies, not in the existence of the error-term $O(x^{-\sigma})$, which is small enough for most purposes, but in the fact that x is greater than a certain multiple of t. Thus the finite sum by which we represent $\zeta(s)$ contains more than At terms. We shall now obtain a formula, which is more complicated than 2.12 (1), but in which the finite sums contain a much smaller number of terms.

The restriction on x arises from the inequality which we obtain for the integrals in § 2.12 (2). Now we can modify the argument applied, e.g., to the second of these integrals, by writing

$$(1) \qquad -\cot \pi z - i = 2i \sum_{\nu \leqq n} e^{2\nu\pi i z} + 2i\, e^{2(n+1)z\pi i}/(1 - e^{2z\pi i}).$$

Proceeding as before, the last term in (1) leads to an error-term

$$O\left\{x^{-\sigma}\int_0^\infty e^{-2(n+1)\pi r - |t|r/x}\,dr\right\} = O\left\{\frac{x^{-\sigma}}{2(n+1)\pi - |t|/x}\right\},$$

and this is $O(x^{-\sigma})$ if

$$(2) \qquad 2(n+1)\pi - |t|/x > A,$$

i.e. for comparatively small values of x, if n is large. We have now to consider the other terms on the right of (1). Now

$$\int_x^{x+i\infty} e^{2\nu\pi i z} z^{-s}\,dz = \int_0^{i\infty} - \int_0^x = (2\nu\pi)^{s-1} e^{\frac{1}{2}\pi i(s-1)}\,\Gamma(1-s) - \int_0^x.$$

Dealing with the other integral in the same way, we obtain altogether

$$(3) \quad 2^s \pi^{s-1} \sin \tfrac{1}{2}s\pi\, \Gamma(1-s) \sum_{\nu=1}^{n} \nu^{s-1} - \sum_{\nu=1}^{n} \int_0^x (e^{2\nu\pi i z} + e^{-2\nu\pi i z})\, z^{-s}\,dz.$$

The first sum is precisely the beginning of the series for $\zeta(s)$, valid for $\sigma < 0$, which we deduce from the functional equation. It is therefore suggested that we can obtain a more useful approximation to $\zeta(s)$ in the critical strip by taking a number of terms of the series valid for $\sigma < 0$, as well as terms from the series valid for $\sigma > 1$. The result, known as 'the approximate functional equation,' is as follows:

Theorem 22. *If* $2\pi xy = |t|$, *and* $\chi = \chi(s) = 2^s \pi^{s-1} \sin \frac{1}{2}\pi s\,\Gamma(1-s)$, *and* h *and* k *are positive constants, then*

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi \sum_{n<y} n^{s-1} + O(x^{-\sigma}) + O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma})$$

uniformly for $-h \leqq \sigma \leqq h$, $x > k$, $y > k$.

* Hardy and Littlewood (3), (4) and (6).

Notice that $$|\chi(s)| = \left\{1 + O\left(\frac{1}{t}\right)\right\}\left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma},$$
and that the last term in each sum is of the order of one of the error-terms.

We give the proof of the case where σ has a fixed value between 0 and 1. In view of our previous remarks it is only a question of dealing with the second sum in (3). We suppose first that x is half an odd integer, and that $x \geqq y$, so that $x \geqq \sqrt{(|t|/2\pi)}$. We take the number n which occurs in (1), (2), and (3) to be $[y]+1$. Then $y < n \leqq y+1$, and condition (2) is satisfied. We suppose also that $t > 0$.

In the first $n-2$ terms of the sum in question we replace the integral along $(0, x)$ by integrals along $(0, -ix)$ and $(-ix, x)$. In the first part we have $z^{-s} = |z|^{-\sigma} e^{-\frac{1}{2}\pi t}$, and these terms are
$$O\left\{\sum_{\nu=1}^{n-2} \int_0^x e^{2\nu x\pi} e^{-\frac{1}{2}\pi t} r^{-\sigma}\, dr\right\} = O(e^{-\frac{1}{2}\pi t + 2nx\pi} n x^{1-\sigma}) = O(e^{-At}).$$
In the second we have
$$z = x - re^{\frac{1}{4}i\pi} \qquad (0 \leqq r \leqq x\sqrt{2}),$$
and $$|z^{-s}| = (x^2 - xr\sqrt{2} + r^2)^{-\frac{1}{2}\sigma} \exp\left(-t \arctan \frac{r}{x\sqrt{2} - r}\right)$$
$$= O\{x^{-\sigma} \exp(-tr/x\sqrt{2})\} = O\{x^{-\sigma}\exp(-\pi y r\sqrt{2})\},$$
since $$\arctan\{\xi/(1-\xi)\} \geqq \xi \qquad (0 \leqq \xi \leqq 1).$$
Also $$\sum_{\nu=1}^{n-2} e^{2\nu z\pi i} = \frac{e^{2z\pi i} - e^{2(n-1)z\pi i}}{1 - e^{2z\pi i}} = O(1) + O(e^{(n-2)\pi r\sqrt{2}}),$$
so that this integral is
$$O(x^{-\sigma}) \int_0^{x\sqrt{2}} (e^{-\pi y r\sqrt{2}} + e^{(n-2-y)\pi r\sqrt{2}})\, dr = O(x^{-\sigma}).$$

The terms involving $e^{-2\nu z\pi i}$ may be dealt with in the same way. The terms $\nu = n-1$, $\nu = n$ require special consideration. The former gives a term $O(e^{-At})$ as before, together with an integral
$$O\left[x^{-\sigma}\int_0^{x\sqrt{2}} \exp\left\{-t \arctan \frac{r}{x\sqrt{2}-r} + (n-1)\pi r\sqrt{2}\right\} dr\right].$$
Since $(n-1)\pi \leqq y\pi = \frac{1}{2}t/x$, this is
$$O\left[x^{-\sigma}\int_0^{x\sqrt{2}} \exp\left\{-t \arctan \frac{r}{x\sqrt{2}-r} + \frac{t}{\sqrt{2}x} r\right\} dr\right].$$
Writing $\xi = r/x\sqrt{2}$, and observing that
$$-\arctan \frac{\xi}{1-\xi} + \xi \leqq -A\xi^2 \qquad (0 \leqq \xi \leqq 1),$$
we obtain $$O\left\{x^{-\sigma}\int_0^{x\sqrt{2}} e^{-Ar^2 t/x^2}\, dr\right\} = O(x^{1-\sigma} t^{-\frac{1}{2}}).$$

For the term $\nu = n$, we return, with error $O(|\chi| y^{\sigma-1}) = O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1})$, to the original integral along $(x, x+i\infty)$, and obtain

$$O\left(x^{-\sigma}\int_0^\infty e^{-2n\pi r + t\arctan r/x}\,dr\right).$$

Since $2n\pi > 2y\pi = t/x$, this is

$$O\left\{x^{-\sigma}\int_0^\infty e^{-tr/x + t\arctan r/x}\,dr\right\} = O\left(x^{-\sigma}\int_0^\infty e^{-Atr^2/x^2}\,dr\right) = O(x^{1-\sigma}t^{-\frac{1}{2}}).$$

This gives the desired result, under the restrictions stated.

We have finally to remove the restrictions (i) $x \equiv \frac{1}{2} \pmod 1$, (ii) $x \geqq y$. Firstly let $x' = [x] + \frac{1}{2}$, $y' = t/2\pi x'$. Then

$$y - y' = \frac{t}{2\pi x} - \frac{t}{2\pi x'} = O\left(\frac{t}{x^2}\right) = O(1).$$

Hence $$\chi \sum_{n<y} - \chi \sum_{n<y'} = O(|\chi| y^{\sigma-1}) = O(t^{\frac{1}{2}-\sigma} y^{\sigma-1}),$$

whence the theorem follows for any x.

To remove the second restriction, take the result with $x \geqq y$, and change s into $1-s$. We obtain

$$\zeta(1-s) = \sum_{n<x} n^{s-1} + \chi_1 \sum_{n<y} n^{-s} + O(x^{\sigma-1}) + O(y^{-\sigma}|t|^{\sigma-\frac{1}{2}}),$$

where

$$\chi_1 = 2(2\pi)^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s), \quad \chi\chi_1 = 1, \quad \zeta(1-s) = \chi_1\zeta(s).$$

Hence, dividing by χ_1, and noting that $|\chi_1| \sim A\,|t|^{\sigma-\frac{1}{2}}$, we obtain

$$\zeta(s) = \sum_{n<y} n^{-s} + \chi \sum_{n<x} n^{s-1} + O(y^{-\sigma}) + O(x^{\sigma-1}|t|^{\frac{1}{2}-\sigma}),$$

which is the theorem with x and y interchanged. This completes the proof. Slight modifications shew that the result holds for $\sigma = 1$ also.

As an example of the superiority of Theorem 22 over Theorem 19, notice that we can deduce at once from Theorem 22, on taking $x = y = (t/2\pi)^{\frac{1}{2}}$, $\sigma = \frac{1}{2}$, and observing that $\chi(\frac{1}{2}+it) = O(1)$, that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{4}}).$$

2.22. THEOREM 23*. *As* $T \to \infty$,

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim T\log T.$$

We take $\sigma = \frac{1}{2}$, $t > 2$, and $x = t/(2\pi\sqrt{\log t})$, $y = \sqrt{\log t}$.

* Hardy and Littlewood (2) and (4).

Then, since $\chi(\frac{1}{2}+it)=O(1)$,

$$\zeta(s)=\sum_{n<x} n^{-s}+O\Big(\sum_{n<y} n^{-\frac{1}{2}}\Big)+O(t^{-\frac{1}{4}})+O(\log^{-\frac{1}{2}} t)$$

$$=\sum_{n<x} n^{-s}+O(\log^{\frac{1}{4}} t)=Z+O(\log^{\frac{1}{4}} t),$$

say. Since

$$\int_2^T (\log^{\frac{1}{4}} t)^2\,dt=O(T\log^{\frac{1}{2}} T)=o(T\log T),$$

it is, as in the proof of Theorem 19, sufficient to prove that

$$\int_0^T |Z|^2\,dt \sim T\log T.$$

Now
$$\int_0^T |Z|^2\,dt=\int_0^T \sum_{m<x} m^{-\frac{1}{2}-it}\sum_{n<x} n^{-\frac{1}{2}+it}\,dt.$$

In inverting the order of integration and summation, it must be remembered that x is a function of t. The term in (m, n) occurs if

$$x>\operatorname{Max}(m,n)=T_1/\{2\pi\surd(\log T_1)\},$$

say, where $T_1=T_1(m,n)$. Hence, writing $X=T/\{2\pi\surd(\log T)\}$,

$$\int_0^T |Z|^2 dt=\sum\sum_{m,n<X}\int_{T_1}^T m^{-\frac{1}{2}-it}\,n^{-\frac{1}{2}+it}\,dt$$

$$=\sum_{n<X}\frac{T-T_1(n,n)}{n}+\sum\sum'_{m,n<X}\frac{1}{\surd(mn)}\int_{T_1}^T\left(\frac{n}{m}\right)^{it}dt$$

$$=T\sum_{n<X}\frac{1}{n}+O\left(\sum_{n<X}\frac{T_1(n,n)}{n}\right)+O\left(\sum\sum_{m<n<X}\frac{1}{\surd(mn)\log n/m}\right).$$

The first term is

$$T\log X+O(T)=T\log T+o(T\log T).$$

The second term is

$$O\Big(\sum_{n<X}\surd\log n\Big)=O(X\surd\log X)=O(T),$$

and, by the lemma of § 2.13, the last term is

$$O(X\log X)=O(T\surd\log T).$$

This proves the theorem.

2.23. The investigation of mean value formulae of this kind has been carried a good deal further by Littlewood and Ingham*. It is known that if α and β are fixed, $\alpha>\frac{1}{2}$, $\beta>-\frac{1}{2}$, $\alpha+\beta>1$, then

$$\int_1^T \zeta^{(\mu)}(\alpha+it)\,\zeta^{(\nu)}(\beta-it)\,dt\sim\zeta^{(\mu+\nu)}(\alpha+\beta)\,T.$$

* Littlewood (2), Ingham (1).

This formula has been proved with a comparatively small error-term, and, with necessary modifications of the expression on the right, extended to all (even complex) values of α and β. In particular

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T - (1+\log 2\pi - 2\gamma)\,T + O(T^{\frac{1}{2}}\log T),$$

where γ is Euler's constant.

2.31*. In view of the difficulty of proving the approximate functional equation, it is of some interest to have an independent proof of theorems of the type of Theorem 23. The formal basis of this method is a well-known formula in the theory of Fourier integrals. We may write Fourier's reciprocal formulae in the form

$$(1)\quad G(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} F(t)\,e^{ixt}\,dt, \qquad F(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} G(t)\,e^{-ixt}dt.$$

If we multiply the first of these equations by $\bar{G}(x)$ and integrate with respect to x (inverting the order of the integrations on the right), we obtain formally

$$(2)\qquad \int_{-\infty}^{\infty} |G(x)|^2\,dx = \int_{-\infty}^{\infty} |F(x)|^2\,dx.$$

In the cases which we actually use, this process is easily justified. We may assume that both $F(x)$ and $G(x)$ are analytic functions, regular and bounded on the real axis, and absolutely integrable over $(-\infty, \infty)$. The inversion of the order of the integrations is then justified by the absolute convergence of the associated double integral.

Suppose now that $f(s) = \Sigma\, a_n n^{-s}$ is regular and $O(t^A)$ for $\sigma > 0$, except possibly for a pole at $s=1$, and that the series is absolutely convergent for $\sigma > 1$. Then, if $\mathbf{R}(x) > 0$,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s) f(s)\, x^{-s}\,ds = \sum_{n=1}^{\infty} \frac{a_n}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)\,(nx)^{-s}\,ds = \sum_{n=1}^{\infty} a_n e^{-nx}.$$

Now move the contour to $\sigma = \alpha$ $(0 < \alpha < 1)$. Let $R(x)$ be the residue at $s=1$. Then

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(s) f(s)\, x^{-s}\,ds = \sum_{n=1}^{\infty} a_n e^{-nx} - R(x) = \phi(x),$$

say. Putting $x = ie^{\xi - i\delta}$ $(0 < \delta < \frac{1}{2}\pi)$, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\alpha+it)\, f(\alpha+it)\, e^{-i(\frac{1}{2}\pi-\delta)(\alpha+it)}\, e^{-i\xi t}\,dt = e^{\alpha\xi}\,\phi(ie^{\xi-i\delta}).$$

* Titchmarsh (1).

We may therefore put

$$F(\xi)=e^{a\xi}\phi(ie^{\xi-i\delta}),\quad G(t)=(2\pi)^{-\frac{1}{2}}\Gamma(a+it)f(a+it)e^{-i(\frac{1}{2}\pi-\delta)(a+it)},$$

and (2) gives

$$(3)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}|\Gamma(a+it)f(a+it)|^2 e^{(\pi-2\delta)t}\,dt=\int_{-\infty}^{\infty}|\phi(ie^{\xi-i\delta})|^2 e^{2a\xi}\,d\xi$$

$$=\int_0^{\infty}|\phi(ie^{-i\delta}u)|^2 u^{2a-1}\,du.$$

Since $\Gamma(a+it)\sim e^{-\frac{1}{2}\pi|t|}|t|^{a-\frac{1}{2}}\sqrt{(2\pi)}$ as $|t|\to\infty$, the part of the t-integral over $(-\infty, 0)$ is bounded as $\delta\to 0$, and we obtain

$$(4)\quad \int_0^{\infty}|f(a+it)|^2 t^{2a-1}e^{-2\delta t}\,dt=\int_0^{\infty}|\phi(ie^{-i\delta}u)|^2u^{2a-1}\,du+O(1).$$

2.32. We now use 2.31 (4) to prove the following theorem.

THEOREM 23 A*. *As* $\delta\to 0$,

$$(1)\qquad \int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^2e^{-\delta t}\,dt\sim\frac{1}{\delta}\log\frac{1}{\delta}.$$

We take $f(s)=\zeta(s)$, $a=\frac{1}{2}$, in 2.31 (4). Then

$$(2)\quad \int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^2e^{-2\delta t}\,dt=\int_0^{\infty}\left|\frac{1}{\exp(iue^{-i\delta})-1}-\frac{1}{iue^{-i\delta}}\right|^2du+O(1).$$

The u-integrand is bounded over $(0, \pi)$ uniformly in δ, so that this part of the integral is bounded. In the remaining part we express the squared modulus as a product of conjugates, and obtain

$$(3)\quad \int_{\pi}^{\infty}\frac{du}{\{\exp(iue^{-i\delta})-1\}\{\exp(-iue^{i\delta})-1\}}$$

$$+2\mathbf{R}\left\{ie^{i\delta}\int_{\pi}^{\infty}\frac{1}{\exp(-iue^{i\delta})-1}\frac{du}{u}\right\}+\int_{\pi}^{\infty}\frac{du}{u^2}.$$

We turn the line of integration in the first two integrals to $(\pi, \pi+i\infty)$, and use the theorem of residues—a process which is easily justified. The first term becomes

$$(4)\quad 2i\pi\sum_{n=1}^{\infty}\frac{1}{ie^{-i\delta}}\frac{1}{\exp(-2in\pi e^{2i\delta})-1}$$

$$+\int_0^{\infty}\frac{dv}{[\exp\{(i\pi-v)e^{i\delta}\}-1]\,[\exp\{(-i\pi+v)e^{i\delta}\}-1]}.$$

The series is the important part. It is asymptotic to

$$2\pi e^{i\delta}\sum_{n=1}^{\infty}\frac{1}{\exp(2n\pi\sin 2\delta)-1},$$

* The result is equivalent to Theorem 23 in the sense that either can be deduced from the other without further reference to the properties of $\zeta(s)$. See Hardy and Littlewood (2), p. 152.

the absolute value of the difference being less than

$$2\pi \sum_{n=1}^{\infty} \exp(2n\pi \sin 2\delta)\,|1-\exp(4in\pi \sin^2\delta)|\,\{\exp(2n\pi\sin 2\delta)-1\}^{-2}$$
$$= \sum_{n \leqq 1/\delta} O(n^{-1}) + \sum_{n>1/\delta} O(e^{-2n\pi \sin 2\delta}) = O(\log 1/\delta) + O(1/\delta) = O(1/\delta).$$

Also

$$\sum_{n=1} \frac{1}{\exp(2n\pi\sin 2\delta)-1} \sim \int_1^{\infty} \frac{du}{\exp(2u\pi\sin 2\delta)-1} = \frac{1}{2\pi\sin 2\delta}\int_{2\pi \sin 2\delta}^{\infty} \frac{dv}{e^v-1}$$
$$= \frac{1}{2\pi\sin 2\delta}\left\{\int_{2\pi\sin 2\delta}^{1} \frac{dv}{v} + O(1)\right\} \sim \frac{1}{2\pi \sin 2\delta}\log\frac{1}{\delta}.$$

We have still to consider the remaining integrals in (3) and (4). We have

$$|\exp\{(-i\pi+v)\,e^{i\delta}\}-1\,| \geqq \exp(v\cos\delta+\pi\sin\delta)-1,$$

so that the middle term in (3) is

$$O\left\{\int_0^{\infty} \frac{dv}{\exp(v\cos\delta+\pi\sin\delta)-1}\right\} = O\left(\int_{\pi\sin\delta}^{\infty}\frac{dw}{e^w-1}\right) = O\left(\log\frac{1}{\delta}\right).$$

Also $|\exp\{(i\pi - v)\,e^{-i\delta}\}-1\,| > A$; for this is plainly true when $\delta = 0$, and so by continuity for $0 < v < \pi$ and δ small enough; and if $v > \pi$ the modulus of the exponential term is less than 1. So we find in the same way that the last term in (4) is $O(\log 1/\delta)$. This completes the proof.

2.41. The fourth power of $|\zeta(s)|$ yields formulae similar to those involving the square, but they are distinctly more difficult to prove. The interest of dealing with this and higher powers will appear more clearly in Chapter VI. The formulae also have applications in the theory of numbers.

Theorem 24*. *As* $T \to \infty$,

$$(1) \qquad \int_1^T |\zeta(\sigma+it)|^4\,dt \sim \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}\,T \qquad (\sigma > \tfrac{1}{2}).$$

If $\sigma > 1$ we have $\qquad \zeta^2(s) = \Sigma\, d(n)\, n^{-s},$

$d(n)$ denoting, as usual, the number of divisors of n, and we deduce at once from the lemma of § 1.23 that

$$\int_1^T |\zeta(\sigma+it)|^4\,dt = \int_1^T |\Sigma\, d(n)\, n^{-s}|^2\,dt \sim T\,\Sigma\, d^2(n)\, n^{-2\sigma}.$$

The last series was summed by Ramanujan†. If the expression of n in prime factors is $p_1^{m_1}\ldots p_r^{m_r}$, then $d(n) = (m_1+1)\ldots(m_r+1)$. It is easily seen from this that we have the identity

$$\Sigma\, d^2(n)\, n^{-s} = \prod_p \sum_{m=0}^{\infty} (m+1)^2 p^{-ms}.$$

* Hardy and Littlewood (4). † Ramanujan (2), B. M. Wilson (1).

Since

$$\sum_{m=0}^{\infty} (m+1)^2 x^m = (1+x)(1-x)^{-3} = (1-x^2)(1-x)^{-4},$$

it follows that

$$(2) \qquad \Sigma\, d^2(n)\, n^{-s} = \prod_p (1-p^{-2s})(1-p^{-s})^4 = \zeta^4(s)/\zeta(2s).$$

This gives the result stated when $\sigma > 1$.

2.42. The general case may be deduced from the approximate functional equation. We take $x = y$, and follow a method analogous to the proof of Theorem 20. We shall however give a different proof, due to Carlson*, which is independent of the approximate functional equation.

We start from the formula

$$(1) \qquad \Sigma\, d(n)\, n^{-s} e^{-\delta n} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(w-s)\, \zeta^2(w)\, \delta^{s-w}\, dw,$$

obtained by inserting the series $\Sigma\, d(n)\, n^{-w}$ for $\zeta^2(w)$ and integrating term-by-term. Moving the contour to $\mathbf{R}(w) = \alpha$, where $\sigma - 1 < \alpha < \sigma$, we pass the pole of $\Gamma(w-s)$ at $w = s$, with residue $\zeta^2(s)$, and the pole of $\zeta^2(w)$ at $w = 1$, with a residue of the form

$$A\,\Gamma(1-s)\,\delta^{s-1} + A\,\delta^{s-1}\{\Gamma(1-s)\log 1/\delta + \Gamma'(1-s)\} = O(e^{-A|t|}\,\delta^{-A}).$$

If $\delta > |t|^{-A}$, this is $O(e^{-A|t|})$, and we obtain

$$(2) \quad \zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} e^{-\delta n} - \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(w-s)\, \zeta^2(w)\, \delta^{s-w}\, dw + O(e^{-A|t|}).$$

Let us call the first two terms on the right Z_1 and Z_2. Then, as in the proof of Theorem 20,

$$\int_{\frac{1}{2}T}^{T} |Z_1|^2\, dt = O\left(T \Sigma \frac{d^2(n)}{n^{2\sigma}} e^{-2\delta n}\right) + O\left(\Sigma\Sigma' \frac{d(m)\, d(n)\, e^{-(m+n)\delta}}{m^\sigma n^\sigma |\log m/n|}\right)$$
$$= O(T) + O(\delta^{2\sigma-2-\epsilon}),$$

since† $d(n) = O(n^\epsilon)$. Also, putting $w = \alpha + iv$,

$$|Z_2| \leqq \frac{\delta^{\sigma-\alpha}}{2\pi}\left\{\int_{-\infty}^{\infty} |\Gamma(w-s)|\, dv \int_{-\infty}^{\infty} |\Gamma(w-s)\, \zeta^4(w)|\, dv\right\}^{\frac{1}{2}}.$$

Now it follows from the asymptotic formula for $\Gamma(z)$ that the first integral is $O(1)$, while

$$\left\{\int_{-\infty}^{-2T} + \int_{2T}^{\infty}\right\} |\Gamma(w-s)\, \zeta^4(w)|\, dv = O(e^{-AT})$$

for $|t| < T$, since $\qquad \zeta(\alpha + iv) = O(|v|^A)$.

* Carlson (3). † *Handbuch*, § 60.

Hence

$$\int_{\frac{1}{2}T}^{T} |Z_2|^2\,dt = O\left\{\delta^{2(\sigma-\alpha)}\int_{-2T}^{2T} |\zeta^4(w)|\,dv \int_{\frac{1}{2}T}^{T} |\Gamma(w-s)|\,dt\right\} + O(\delta^{2(\sigma-\alpha)})$$

$$= O\left\{\delta^{2(\sigma-\alpha)}\int_{-2T}^{2T} |\zeta^4(\alpha+iv)|\,dv\right\}$$

$$= O\{\delta^{2(\sigma-\alpha)}\,T^{\lambda(\alpha)+\epsilon}\},$$

where $\lambda(\sigma)$ is the lower bound of numbers ξ such that the left-hand side of 2.41 (1) is $O(T^{\xi})$. Hence

(3) $$\int_{\frac{1}{2}T}^{T} |\zeta(s)|^4\,dt = O(T) + O(\delta^{2\sigma-2-\epsilon}) + O(\delta^{2\sigma-2\alpha}\,T^{\lambda(\alpha)+\epsilon}).$$

Choosing δ so that $T^{\lambda(\alpha)} = \delta^{2\alpha-2}$, we see that the right-hand side of (2) is $O(T)$ if $\sigma > 1-(1-\alpha)/\lambda(\alpha)$. For such values of σ, replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$, ... and adding, we obtain

(4) $$\int_{1}^{T} |\zeta(s)|^4\,dt = O(T).$$

Let c be the lower bound of positive numbers σ for which (4) holds. Then we have proved that

(5) $$c \leqq 1-(1-\alpha)/\lambda(\alpha)$$

for all values of α in the range $(0, c)$. But, taking $\sigma = 1-c$ in 1.51 (1), we see that

$$\lambda(1-c) \leqq 4c-2+\lambda(c) = 4c-1.$$

Since $\lambda(1-c) \geqq 1$, it follows that $c \geqq \frac{1}{2}$. Also, putting $\alpha = 1-c$ in (5), we obtain $c \leqq 1-c/(4c-1)$, whence

$$(2c-1)^2 \leqq 0, \quad c = \tfrac{1}{2}.$$

This proves the O-result corresponding to 2.41 (1) for $\sigma > \frac{1}{2}$.

But, by a general theorem on Dirichlet series,* if the O-result holds for $\sigma > \frac{1}{2}$, so does the asymptotic equality stated in the theorem. As this part of the argument has no special connection with the zeta-function, we must refer the reader to Carlson's paper for the proof.

2.43. The corresponding result for $\sigma = \frac{1}{2}$ is

THEOREM 25. *As* $T \to \infty$,

$$\int_{1}^{T} |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \frac{1}{2\pi^2}\,T\log^4 T.$$

In this case even the approximate functional equation merely gives $O(T\log^4 T)$, and a further argument is required. Two proofs of the

* Carlson (2).

result are known*. We may use the method of §§ 2.31, 2.32; or we may use an approximate functional equation for the square of $\zeta(s)$, which runs as follows †:

THEOREM 26. *If* $-\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, $x > A$, $y > A$, $xy = (t/2\pi)^2$, *then*

$$\zeta^2(s) = \sum_{n \leqq x} \frac{d(n)}{n^s} + \chi^2(s) \sum_{n \leqq y} \frac{d(n)}{n^{1-s}} + O\left\{x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{t}\right)^{\frac{1}{4}} \log t\right\}.$$

2.51. We now pass to still higher powers of $\zeta(s)$. In the general case our knowledge is very incomplete, and we can state a mean value formula in a certain restricted range of values of σ only. The two following theorems include all that is known on this point.

THEOREM 27. *For every positive integer* $k > 2$,

$$(1) \qquad \int_1^T |\zeta(\sigma + it)|^{2k} dt \sim T \sum d_k^2(n) n^{-2\sigma}$$

for $\sigma > 1 - 1/k$, $d_k(n)$ *being the number of ways of expressing* n *as a product of* k *factors.*

This may be proved by a straightforward extension of Carlson's method, or by means of the approximate functional equation. We leave the proof to the reader.

THEOREM 28. *If* $\zeta(\frac{1}{2} + it) = O(t^{\lambda+\epsilon})$, *then* (1) *holds for*

$$\sigma > (4k\lambda + 1)/(4k\lambda + 2).$$

We use the following obvious extension of § 2.42 (2):

$$(2) \quad \zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} e^{-\delta n} - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(w-s)\, \zeta^k(w)\, \delta^{s-w}\, dw + O(e^{-A|t|}).$$

Taking $a = \frac{1}{2}$ we obtain

$$(3) \qquad \zeta^k(s) = \sum d_k(n)\, n^{-s} e^{-\delta n} + O(\delta^{\sigma - \frac{1}{2}} t^{(\lambda+\epsilon)k}).$$

Now, as in § 2.42,

$$\int_T^{2T} |\sum d_k(n)\, n^{-s} e^{-\delta n}|^2 dt = T \sum d_k^2(n)\, n^{-2\sigma} e^{-2\delta n} + O(\delta^{2\sigma-2-\epsilon}),$$

and the O-term is negligible if $\delta^{2\sigma-2-2\epsilon} = T$.

* Ingham (1), Titchmarsh (1). † Hardy and Littlewood (6).

The O-term in (3) is then $o(1)$ if

$$\frac{\sigma-\frac{1}{2}}{2\sigma-2}+\lambda k<0,$$

i.e. if

$$\sigma>\frac{4k\lambda+1}{4k\lambda+2}.$$

The result follows as in previous cases.

2.52. If we use Theorem 15, so that we may take $\lambda=\frac{1}{6}$, we obtain a more extended range from Theorem 28 than from Theorem 27 if $k>6$. Theorem 16 would of course extend the range still further. It is quite possible that $\zeta(\frac{1}{2}+it)=O(t^{\epsilon})$, in which case we can put $\lambda=0$, and the result holds for $\sigma>\frac{1}{2}$, for every k. In the present state of knowledge we cannot extend the range of validity of 2.51 (1) to $\sigma>\frac{1}{2}$ for any value of k greater than 2. Nor is any mean value formula for

$$|\zeta(\tfrac{1}{2}+it)|^{2k}$$

known for $k>2$.

2.53. On the other hand we can, in this case, obtain a large *lower* bound for the mean value for every value of k.

THEOREM 29 *. $\displaystyle\int_0^T|\zeta(\tfrac{1}{2}+it)|^{2k}\,dt>C_kT(\log T)^{k^2}$,

where C_k depends on k only.

Putting

$$f(s)=\{\zeta(s)\}^k,\quad a=\tfrac{1}{2}$$

in § 2.31 (4), we obtain

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt=\int_0^{\infty}|\Sigma d_k(n)\exp(-ie^{-i\delta}nu)-R(u)|^2\,du+O(1),$$

where $R(u)$ is the residue term. The integral on the right falls into two parts which behave very differently. For $u>1$ the term $R(u)$ is trivial and may be omitted; and we can approximate to the integral over $(1,\infty)$ by expressing the integrand as a product of conjugates and integrating term-by-term. We find that this part is greater than a constant multiple of $(1/\delta)\log^{k^2}(1/\delta)$. On the other hand it is, for $k>2$, very difficult to prove anything about the integral over $(0, 1)$; its very existence depends on some sort of cancelling between the large parts of the series and of the residue term. But the integrand is positive, and so, in the problem of the lower bound, we may omit this part altogether.

* Titchmarsh (4).

It is an argument of the same kind which enables us to prove Theorem 17. It follows at once from Theorem 29 that

$$\zeta(\tfrac{1}{2}+it)=\Omega(\log^{\frac{1}{2}k} t)$$

for every k; and we may expect to obtain a still better result by making k a function of δ. Actually we apply the argument, not to $\zeta(\frac{1}{2}+it)$ directly, but to $\zeta(\sigma+it)$ with $\sigma>\frac{1}{2}$, and then deduce the case $\sigma=\frac{1}{2}$ from the Phragmén-Lindelöf theorem.

CHAPTER III

THE DISTRIBUTION OF THE ZEROS

3.1. *The Riemann hypothesis.*

3.11. The paper in which Riemann first considered the zeta-function has become famous for the number of ideas it contains which have since proved fruitful, and it is by no means certain that its riches are even now exhausted. The analysis which precedes his observations on the zeros is particularly interesting. He writes

$$(1)\qquad \sum_{n=1}^{\infty} e^{-n^2\pi x}=\psi(x),$$

and obtains, by term-by-term integration, the formula

$$(2)\qquad \Gamma(\tfrac{1}{2}s)\,\pi^{-\frac{1}{2}s}\zeta(s)=\int_0^\infty \psi(x)\,x^{\frac{1}{2}s-1}\,dx.$$

Using the functional equation*

$$(3)\qquad 2\psi(x)+1=x^{-\frac{1}{2}}\{2\psi(1/x)+1\},$$

we see that the right-hand side of (2) is equal to

$$\int_1^\infty \psi(x)\,x^{\frac{1}{2}s-1}\,dx+\int_0^1 \psi(1/x)\,x^{\frac{1}{2}(s-3)}\,dx+\tfrac{1}{2}\int_0^1 (x^{\frac{1}{2}(s-3)}-x^{\frac{1}{2}s-1})\,dx$$

$$=\frac{1}{s(s-1)}+\int_1^\infty \psi(x)\{x^{\frac{1}{2}s-1}+x^{-\frac{1}{2}(1+s)}\}\,dx.$$

Putting $s=\frac{1}{2}+it$, and writing†

$$\Xi(t)=\xi(s)=\tfrac{1}{2}s(s-1)\,\Gamma(\tfrac{1}{2}s)\,\pi^{-\frac{1}{2}s}\zeta(s),$$

we obtain the formula

$$(4)\qquad \Xi(t)=\tfrac{1}{2}-(t^2+\tfrac{1}{4})\int_1^\infty \psi(x)\,x^{-\frac{3}{4}}\cos(\tfrac{1}{2}t\log x)\,dx.$$

* Whittaker and Watson, *Modern Analysis*, (ed. 4, 1927), Ch. VI, ex. 17.

† Following Landau, we use Ξ where Riemann uses ξ.

If we integrate by parts and use the relation $4\psi'(1)+\psi(1)=-\frac{1}{2}$, which follows at once from (3), we obtain

$$(5) \qquad \Xi(t)=4\int_1^\infty \frac{d}{dx}\{x^{\frac{3}{2}}\psi'(x)\}x^{-\frac{1}{4}}\cos(\tfrac{1}{2}t\log x)\,dx.$$

Riemann then observes, 'Diese Function ist für alle endlichen Werthe von t endlich, und lässt sich nach Potenzen von tt in eine sehr schnell convergirende Reihe entwickeln. Da für einen Werth von s, dessen reeller Bestandtheil grösser als 1 ist, $\log\zeta(s)=-\Sigma\log(1-p^{-s})$ endlich bleibt, und von den Logarithmen der übrigen Factoren von $\Xi(t)$ dasselbe gilt, so kann die Function $\Xi(t)$ nur verschwinden, wenn der imaginäre Theil von t zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ liegt. Die Anzahl der Wurzeln von $\Xi(t)=0$, deren reeller Theil zwischen 0 und T liegt, ist etwa

$$=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi};$$

denn dass Integral $\int d\log\Xi(t)$ positive um den Inbegriff der Werthe von t erstreckt, deren imaginäre Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$, und deren reeller Theil zwischen 0 und T liegt, ist $\Big($bis auf einen Bruchtheil von der Ordnung der Grösse* $\frac{1}{T}\Big)$ gleich $\left(T\log\frac{T}{2\pi}-T\right)i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\Xi(t)=0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reelle sind.'

This statement, that all the zeros of $\Xi(t)$ are real, is the famous 'Riemann hypothesis', which remains unproved to this day. The memoir goes on 'Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung [i.e. the explicit formula for $\pi(x)$] entbehrlich schien.'

The approximate formula for the number of zeros of $\Xi(t)$ with real part between 0 and T has of course been proved since Riemann's time; and something too has been discovered about the real zeros. But one would like to know what Riemann meant by saying that 'we find about as many real zeros as there are zeros altogether.' There seems to be no clue to the exact significance of this remark. Riemann apparently never returned to the subject, and it was 34 years before the researches of

* It is usually supposed that this $1/T$ is a mistake for $\log T$; for, since $N(T)$ has an infinity of discontinuities at least equal to 1, the remainder cannot tend to zero.

Hadamard began to shew the correctness of some of Riemann's conjectures.

3.12. The formula 3.11 (5) is interesting because Riemann seems to have attached considerable importance to it, though he made no definite use of it, except to obtain his 'rapidly convergent series.' It is only recently that Pólya* has used it to prove that $\Xi(t)$ has an infinity of real zeros—a result which had been obtained some time before in another way (see § 3.3). But the formula has some curious features. We can write it in the form

$$\Xi(t) = 2\int_0^\infty \Phi(u)\cos 2ut\,du,$$

where

$$\Phi(u) = 4\sum_{n=1}^{\infty}(2n^4\pi^2 e^{9u} - 3n^2\pi e^{5u})\,e^{-n^2\pi e^{4u}}.$$

This is an even function of u, and is positive for all real values of u. As $u \to \pm\infty$,

$$\Phi(u) \sim 16\pi^2\cosh 9u\, e^{-2\pi\cosh 4u}.$$

If we denote the function on the right-hand side by $\Phi_1(u)$, and put

$$\Xi_1(t) = 2\int_0^\infty \Phi_1(u)\cos 2ut\,du,$$

then $\Xi_1(t)$ *is an integral function of* t, *all of whose zeros are real* †.

A still closer approximation to $\Phi(u)$ is given by

$$\Phi_2(u) = (16\pi^2\cosh 9u - 24\pi\cosh 5u)\,e^{-2\pi\cosh 4u},$$

and this also leads to a function $\Xi_2(t)$, all of whose zeros are real ‡.

It is also possible to deduce from the formula necessary and sufficient conditions for the reality of the roots of $\Xi(t)$. One such condition § is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(\alpha)\Phi(\beta)\,e^{i(\alpha+\beta)x}\,e^{(\alpha-\beta)y}(\alpha-\beta)^2\,d\alpha\,d\beta \geqq 0,$$

for all real values of x and y. But no method has been suggested of shewing whether such criteria are satisfied or not.

3.13. The smallest zeros of $\Xi(t)$ have been calculated with great accuracy. They are

$$\alpha_1 = 14{\cdot}13\ldots,\qquad \alpha_2 = 21{\cdot}02\ldots,\qquad \alpha_3 = 25{\cdot}01\ldots,$$
$$\alpha_4 = 30{\cdot}42\ldots,\qquad \alpha_5 = 32{\cdot}93\ldots,\qquad \alpha_6 = 37{\cdot}58\ldots,$$

and so on. They are all real. It was shewn by Backlund ‖ that the interval $0 < t < 200$ contains 79 real zeros, and that these are the only zeros with $0 < \mathbf{R}(t) < 200$; and Hutchinson ¶ has verified the reality of

* Pólya (3). † Pólya (1), (2). ‡ Pólya (3). § See Pólya (3), § 7.
‖ Backlund (1), (2). ¶ Hutchinson (1).

the roots as far as $t = 300$. Naturally the further we go the more complicated the calculations become. We shall content ourselves with shewing that the first zero, at any rate, is real, and finding its position very roughly. We leave to the reader only some quite trivial numerical calculations.

3.2. THEOREM 30. *The complex zero $\beta + i\gamma$ $(\gamma > 0)$ of $\zeta(s)$ which is nearest to the real axis lies on $\sigma = \frac{1}{2}$, between $t = 10$ and $t = 18$.*

We consider the number $N(T)$ of zeros $\beta + i\gamma$ such that $0 < \gamma \leqq T$. We prove that there are no zeros on the lines $t = 10$, $t = 18$, and then that $N(10) < 1$ and that $0 < N(18) < 2$. Hence $N(10) = 0$, $N(18) = 1$. There is therefore just one zero between $t = 10$ and $t = 18$; and it must lie on $\sigma = \frac{1}{2}$, since zeros not on $\sigma = \frac{1}{2}$ occur in symmetrical pairs.

It follows from § 2.12 that, if x is half an odd integer, and $\sigma \geqq \frac{1}{2}$, then

$$\zeta(s) = \sum_{n<x} n^{-s} + \delta, \tag{1}$$

where

$$|\delta| < \frac{x^{\frac{1}{2}}}{t} + \frac{2}{x^{\frac{1}{2}}(2\pi - |t|/x)}. \tag{2}$$

Putting $t = 10$, $x = \frac{5}{2}$ and taking the real part, we obtain

$$\mathbf{R}\,\zeta(\sigma + 10i) = 1 + 2^{-\sigma}\cos(10\log 2) + \delta',$$

where, as is easily seen from (2) (replacing π by 3), $|\delta'| < 1$. Also $\cos(10\log 2) > 0$. Hence $\mathbf{R}\,\zeta(\sigma + 10i) > 0$ for $\sigma \geqq \frac{1}{2}$, and $\zeta(\sigma + 10i) \neq 0$. Similarly, putting $t = 18$, $x = \frac{9}{2}$, we obtain

$$\mathbf{R}\,\zeta(\sigma + 18i) = 1 + \sum_{n=2}^{4} n^{-\sigma}\cos(18\log n) + \delta'',$$

where $|\delta''| < 1$. Also $\cos(18\log n)$ is positive for $n = 2, 3, 4$. Hence $\mathbf{R}\,\zeta(\sigma + 18i) > 0$ for $\sigma \geqq \frac{1}{2}$, and $\zeta(\sigma + 18i) \neq 0$.

Now

$$\pi N(T) = \Delta\,\{\mathrm{am}\, s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)\}, \tag{3}$$

where Δ denotes the variation along 2, $2 + iT$, $\frac{1}{2} + iT$. It is easily seen that

$$\Delta\ \mathrm{am}\, s(s-1) = \pi, \quad \Delta\ \mathrm{am}\ \pi^{-\frac{1}{2}s} = -\tfrac{1}{2}T\log\pi.$$

Also, from Binet's formula*

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + 2\int_0^\infty \frac{\arctan(u/z)}{e^{2\pi u} - 1}\,du,$$

* Whittaker and Watson, *Modern Analysis* (ed. 4, 1927), §12.32.

it follows that

$$\text{(4)} \qquad \Delta \operatorname{am} \Gamma(\tfrac{1}{2}s) = \mathbf{I} \log \Gamma(\tfrac{1}{4} + \tfrac{1}{2}iT)$$

$$= \tfrac{1}{2}T \log |\tfrac{1}{4} + \tfrac{1}{2}iT| - \tfrac{1}{4} \arctan 2T - \tfrac{1}{2}T - \int_0^\infty \log \left| \frac{u + \frac{1}{2}T - \frac{1}{4}i}{u - \frac{1}{2}T + \frac{1}{4}i} \right| \frac{du}{e^{2\pi u} - 1}.$$

Since the second and fourth terms are negative, we deduce that

$$\text{(5)} \qquad \pi N(T) < \tfrac{1}{2}T \log \left\{ \frac{1}{\pi e} (\tfrac{1}{4}T^2 + \tfrac{1}{16})^{\frac{1}{2}} \right\} + \pi + \Delta \operatorname{am} \zeta(s).$$

Since $\mathbf{R}\zeta(s)$ remains positive on the contour for $T = 10$ or 18, $\Delta \operatorname{am} \zeta(s)$ remains between $\pm \frac{1}{2}\pi$; and we can now easily verify that $N(10) < 1$, $N(18) < 2$.

In the problem of the lower bound we have to take into account the integral in (4). We have

$$\log \left| \frac{u + \frac{1}{2}T - \frac{1}{4}i}{u - \frac{1}{2}T + \frac{1}{4}i} \right| = \tfrac{1}{2} \log \left\{ 1 + \frac{2uT}{(u - \frac{1}{2}T)^2 + \frac{1}{16}} \right\} < \frac{uT}{(\frac{1}{2}T - 1)^2}$$

for $u < 1$, $T > 2$. Since $e^{2\pi u} - 1 > 2\pi u$, the integral over $(0, 1)$ is less than

$$\int_0^1 \frac{uT}{(\frac{1}{2}T - 1)^2} \frac{du}{2\pi u} = \frac{T}{2\pi(\frac{1}{2}T - 1)^2} < \frac{8}{\pi T} \qquad (T > 4).$$

Also $e^{2\pi u} - 1 > 20\pi u$ for $u > 1$, so that the integral over $(1, \infty)$ is less than

$$\int_0^\infty \log \left| \frac{u + \frac{1}{2}T}{u - \frac{1}{2}T} \right| \frac{du}{20\pi u} = \frac{\pi}{40}.$$

Since also $\arctan 2T < \frac{1}{2}\pi$, we obtain

$$\text{(6)} \qquad \pi N(T) > \tfrac{1}{2}T \log \frac{T}{2\pi e} + \frac{17}{20}\pi - \frac{8}{\pi T} + \Delta \operatorname{am} \zeta(s),$$

and $N(18) > 0$ easily follows from this.

3.3. *General theorems on the zeros which lie on $\sigma = \frac{1}{2}$.*

3.31. THEOREM 31*. *There are an infinity of zeros on $\sigma = \frac{1}{2}$.*

Let

$$Z(t) = e^{\frac{1}{4}\pi t} \Xi(t)/(t^2 + \tfrac{1}{4}) = -\tfrac{1}{2}\pi^{-\frac{1}{4} - \frac{1}{2}it} e^{\frac{1}{4}\pi t} \Gamma(\tfrac{1}{4} + \tfrac{1}{2}it) \zeta(\tfrac{1}{2} + it).$$

Then $Z(t)$ is real for real t, and the zeros of $\zeta(s)$ on $\sigma = \frac{1}{2}$ correspond to the real zeros of $Z(t)$. We prove that $Z(t)$ has an infinity of real zeros.

The idea of the proof is to contrast the behaviour of the integrals

$$\int_T^{2T} Z(t)\, dt, \quad \int_T^{2T} |Z(t)|\, dt,$$

* Hardy (1), Landau (4), de la Vallée Poussin (1) (2), Hardy and Littlewood (2), Fekete (1), Pólya (3).

as $T \to \infty$. If $Z(t)$ has only a finite number of zeros, it is ultimately of constant sign, and the second of the integrals is equal to the modulus of the first. We shall see that this assumption leads to a contradiction.

We have, if $\mathbf{R}(x) > 0$,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(\tfrac{1}{2}s)\,\zeta(s)\,x^{-\frac{1}{2}s}\,ds = \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(\tfrac{1}{2}s)\,(n^2x)^{\frac{1}{2}s}\,ds = 2\sum_{1}^{\infty} e^{-n^2x}.$$

Moving the line of integration to $\sigma = \frac{1}{2}$, we pass a pole at $s=1$ with residue $\sqrt{(\pi/x)}$. Hence

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(\tfrac{1}{2}s)\,\zeta(s)\,x^{\frac{1}{2}s}\,ds = 2\sum_{1}^{\infty} e^{-n^2x} - \sqrt{\left(\frac{\pi}{x}\right)} = \phi(x),$$

say; or, in terms of $Z(t)$,

$$(1) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{4}\pi t}\,Z(t)\left(\frac{\pi}{x}\right)^{\frac{1}{4}+\frac{1}{2}it} dt = -\,\phi(x).$$

Putting $x = \pi e^{i(\frac{1}{2}\pi-\delta)}$, where δ is small and positive, and using the fact that $e^{-\frac{1}{4}\pi t}\,Z(t)$ is an even function of t, we obtain

$$(2) \qquad \frac{2}{\pi}\int_{0}^{\infty} \cosh\left\{\left(\frac{\pi}{2}-\delta\right)\frac{t}{2}\right\} e^{-\frac{1}{4}\pi t}\,Z(t)\,dt = \phi\{\pi e^{i(\frac{1}{2}\pi-\delta)}\}$$

$$= O\left(\sum_{1}^{\infty} e^{-n^2\pi\sin\delta}\right) + O(1) = O\left(\int_{0}^{\infty} e^{-u^2\pi\sin\delta}\,du\right) + O(1) = O(\delta^{-\frac{1}{2}}).$$

Hence if $Z(t) \neq 0$ for $t > t_0$, then, for $T > t_0$,

$$\int_{T}^{2T}\left|Z(t)\right|dt = \left|\int_{T}^{2T} Z(t)\,dt\right| < e\left|\int_{T}^{2T} e^{\frac{1}{4}\pi t - \frac{1}{2}t/T}\,e^{-\frac{1}{4}\pi t}\,Z(t)\,dt\right|$$

$$(3) \qquad < 2e\left|\int_{t_0}^{\infty} \cosh\left(\frac{1}{4}\pi t - \frac{1}{2}\frac{t}{T}\right) e^{-\frac{1}{4}\pi t}\,Z(t)\,dt\right| = O(\sqrt{T}).$$

On the other hand

$$|Z(t)| = \left|-\left(\frac{2}{\pi}\right)^{-\frac{1}{4}} e^{-\frac{1}{8}\pi i}\,(2\pi e)^{-\frac{1}{2}it}\,e^{\frac{1}{2}it\log t}\,t^{-\frac{1}{4}}\left\{1+O\left(\frac{1}{t}\right)\right\}\zeta(\tfrac{1}{2}+it)\right|$$

$$> At^{-\frac{1}{4}}|\zeta(\tfrac{1}{2}+it)|,$$

so that

$$\int_{T}^{2T}|Z(t)|\,dt > AT^{-\frac{1}{4}}\int_{T}^{2T}|\zeta(\tfrac{1}{2}+it)|\,dt > AT^{-\frac{1}{4}}\left|\int_{T}^{2T}\zeta(\tfrac{1}{2}+it)\,dt\right|\text{*}.$$

* There is no great difference between $\int_{T}^{2T}|\zeta(\frac{1}{2}+it)|\,dt$ and $\left|\int_{T}^{2T}\zeta(\frac{1}{2}+it)\,dt\right|$, though we doubtless lose something by this step.

But

$$i\int_T^{2T}\zeta(\tfrac{1}{2}+it)\,dt=\int_{\frac{1}{2}+iT}^{\frac{1}{2}+2iT}\zeta(s)\,ds=\int_{\frac{1}{2}+iT}^{2+iT}+\int_{2+iT}^{2+2iT}+\int_{2+2iT}^{\frac{1}{2}+2iT}$$

$$=\left[s-\sum^{\infty}\frac{1}{n^s\log n}\right]_{2+iT}^{2+2iT}+\int_{\frac{1}{2}}^{2}O(\sqrt{T})\,d\sigma=iT+O(\sqrt{T}).$$

Hence

$$(4)\qquad \int_T^{2T}|Z(t)|\,dt>AT^{\frac{3}{4}}.$$

Since this contradicts (3) the theorem is proved.

3.41. We now denote by $N_0(T)$ the number of zeros on $\sigma=\frac{1}{2}$ with ordinate less than T. We have proved in § 3.31 that $N_0(T)\to\infty$ with T. By an elaboration of the same idea we can prove much more than this.

THEOREM 32*. $N_0(T)>AT.$

The proof is based on the same ideas as the previous one, but we consider integrals of a more complicated kind. We contrast the behaviour of the integrals

$$(1)\quad I=\int_t^{t+H}Z(u)\,e^{-u/T}\,du,\quad J=\int_t^{t+H}|Z(u)|\,e^{-u/T}du\quad (T\leqq t\leqq 2T)$$

as $t\to\infty$, H being constant. We begin by proving that

$$(2)\qquad \int_T^{2T}|I|^2\,dt<AH\sqrt{T}\quad (T>T_0=T_0(H)).$$

3.42. We require a development of the Fourier formulae of §2.31. It is easily seen that the functions

$$\int_t^{t+H}G(x)\,dx,\qquad F(x)\,(e^{iHx}-1)/ix$$

are related in the same way as $G(x)$ and $F(x)$, so that

$$(1)\quad \int_{-\infty}^{\infty}\left|\int_t^{t+H}G(x)\,dx\right|^2dt=\int_{-\infty}^{\infty}|F(x)|^2\,\frac{4\sin^2\frac{1}{2}Hx}{x^2}\,dx.$$

Now, putting $x=\pi e^{i(\frac{1}{4}\pi-\delta)+2\xi}$ in 3.31 (1), we see that

$$Z(t)\,e^{-\frac{1}{2}\delta t},\quad -\sqrt{(\tfrac{1}{2}\pi)}\,e^{\frac{1}{2}\xi+\frac{1}{2}i(\frac{1}{4}\pi-\delta)}\,\phi\left(\pi e^{i(\frac{1}{4}\pi-\delta)+2\xi}\right)$$

are related like $G(t)$ and $F(\xi)$. Hence (1) gives

* Hardy and Littlewood (3).

(2)

$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} Z(u)\,e^{-\frac{1}{2}\delta u}\,du\right|^2 dt = 2\pi\int_{-\infty}^{\infty} e^{\xi}\,|\,\phi\,(\pi e^{i(\frac{1}{2}\pi-\delta)+2\xi})\,|^2\,\frac{\sin^2(\frac{1}{2}H\xi)}{\xi^2}\,d\xi$$

$$= 4\pi\int_0^{\infty}|\,\phi\,(\pi e^{i(\frac{1}{2}\pi-\delta)}y)\,|^2\,\frac{\sin^2(\frac{1}{4}H\log y)}{\log^2 y}\frac{dy}{\sqrt{y}}$$

$$= 8\pi\int_1^{\infty}|\,\phi\,(\pi e^{i(\frac{1}{2}\pi-\delta)}\,y)\,|^2\,\frac{\sin^2(\frac{1}{4}H\log y)}{\log^2 y}\frac{dy}{\sqrt{y}},$$

the last form being obtained by using the relation

$$\phi\,(z) = \sqrt{(\pi/z)}\,\phi\,(\pi^2/z),$$

equivalent to 3.11 (3), in the range (0, 1).

The left-hand side of (2) is greater than that of 3.41 (2) if $\delta = 2/T$. We have therefore to prove that the right-hand side of (2) is $O(H\delta^{-\frac{1}{2}})$. Now it is less than

$$\int_1^{\infty}\{A\,|\,\Sigma e^{-n^2\pi y(\sin\delta + i\cos\delta)}\,|^2 + A'\,y^{-1}\}\,\frac{\sin^2(\frac{1}{4}H\log y)}{\log^2 y}\frac{dy}{\sqrt{y}}.$$

In the term involving A' we omit the factor $y^{-\frac{1}{2}}$, and this part is then simply a multiple of H. In the other part we put

$$\sin^2(\tfrac{1}{4}H\log y) < AH^2\log^2 y \quad (y < 1 + 1/H), \ \leqq 1\,(y > 1 + 1/H),$$

and obtain

(3) $$AH^2\int_1^{1+1/H}|\,\Sigma\,|^2\,dy + A\int_{1+1/H}^{\infty}\frac{|\,\Sigma\,|^2}{\log^2 y}\frac{dy}{\sqrt{y}}.$$

Now

(4) $$|\Sigma|^2 = \Sigma e^{-2n^2\pi y\sin\delta} + 2\sum_{n<m}\sum e^{-(m^2+n^2)\pi y\sin\delta + i(m^2-n^2)\pi y\cos\delta}.$$

As in 3.31 (2) the first sum is $O\{(y\delta)^{-\frac{1}{2}}\}$, and its contribution to the integral (3) is therefore

$$O\left\{H^2\int_1^{1+1/H}(y\delta)^{-\frac{1}{2}}dy\right\} + O\left\{\int_{1+1/H}^{\infty}\delta^{-\frac{1}{2}}\frac{dy}{y\log^2 y}\right\} = O\,(H\delta^{-\frac{1}{2}}).$$

The second term in (4) contributes to the second integral in (3) terms of the form

$$\int_{1+1/H}^{\infty} e^{-(m^2+n^2)\pi y\sin\delta + i(m^2-n^2)\pi y\cos\delta}\,\frac{dy}{\sqrt{y}\log^2 y}.$$

Turning the line of integration through $+\frac{1}{2}\pi$, and putting

$$y = 1 + 1/H + ir,$$

we see that this is

$$O\left\{e^{-(m^2+n^2)\pi\sin\delta}\int_0^{\infty} e^{-(m^2-n^2)\pi r\cos\delta}\,H^2\,dr\right\} = O\,\{e^{-(m^2+n^2)\pi\sin\delta}(m^2-n^2)^{-1}H^2\},$$

and therefore the whole series contributes

$$O\left(H^2 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \frac{e^{-(m^2+n^2)\pi \sin \delta}}{m^2-n^2}\right) = O\left(H^2 \sum_{m=2}^{\infty} \frac{e^{-m^2\pi \sin\delta}}{m} \sum_{n=1}^{m-1} \frac{1}{m-n}\right)$$
$$= O\left(H^2 \sum_{m=2}^{\infty} m^{-1} \log m e^{-m^2\pi \sin\delta}\right)$$
$$= O\left(H^2 \sum_{m<1/\delta} m^{-1} \log m + H^2 \sum_{m \geqq 1/\delta} e^{-m^2 \pi \sin \delta}\right)$$
$$= O(H^2 \log^2 1/\delta) = O(H\delta^{-\frac{1}{2}})$$

for $\delta < \delta_0 = \delta_0(H)$. This gives the desired result for the second integral in (3). The first integral may be dealt with in the same way, or, still more simply, by actually integrating each term. This proves 3.41 (2).

3.43. We next prove that

$$(1) \qquad J > (AH + \Psi) T^{-\frac{1}{4}},$$

where

$$(2) \qquad \int_T^{2T} |\Psi|^2 \, dt < AT \qquad (0 < H < T).$$

This corresponds to 3.31 (4). We have, if $s = \frac{1}{2} + it$, $T < t < 2T$,

$$T^{\frac{1}{4}} |Z(t)| > A |\zeta(s)| > A |(1 - 2^{1-s}) \zeta(s)| {}^* = A \left| 1 + \sum_2^{\infty} (-1)^{n-1} n^{-s} \right|$$
$$> A\mathbf{R} \left\{ 1 + \sum_2^{\infty} (-1)^{n-1} n^{-s} \right\},$$

and hence

$$T^{\frac{1}{4}} J > AH + A\mathbf{R} \left[i \sum_2^{\infty} (-1)^{n-1} n^{-s} / \log n \right]_{s=\frac{1}{2}+it}^{\frac{1}{2}+i(t+H)}$$
$$= AH + A\mathbf{R} \left[i\, g(s) \right]_{\frac{1}{2}+it}^{\frac{1}{2}+i(t+H)},$$

say. It is now sufficient to prove that

$$(3) \qquad \int_T^{2T} |g\{\tfrac{1}{2} + i(t+u)\}|^2 \, dt = O(T)$$

uniformly for $0 \leqq u \leqq T$; and this clearly follows from the particular case $u = 0$. Now $g(s)$ is regular and $O(t^A)$, for $\sigma > 0$, so that by 2.31 (4),

$$(4) \quad \int_0^{\infty} |g(\tfrac{1}{2} + it)|^2 e^{-2\delta t} \, dt = \int_0^{\infty} \left| \sum_{n=2}^{\infty} (-1)^{n-1} \frac{e^{-nixe^{i\delta}}}{\log n} \right|^2 dx + O(1).$$

The left-hand side of (4) is greater than a constant multiple of that of (3),

* This step avoids the occurrence in the function $g(s)$ of a pole which brings in residue terms.

with $u=0$, if $\delta=1/T$. On the right of (4), the integrand is bounded for small x and $0<\delta<\frac{1}{2}\pi$, since, putting $w=-\exp(-ixe^{i\delta})$, so that $|1-w|>A$,

$$\left|\sum_2^\infty \frac{w^n}{\log n}\right| = \left|-\frac{w}{\log 2}+\frac{1}{1-w}\sum_2^\infty (w-w^{n+1})\left(\frac{1}{\log n}-\frac{1}{\log n+1}\right)\right|$$
$$<\frac{1}{\log 2}+A\sum_2^\infty 2\left(\frac{1}{\log n}-\frac{1}{\log n+1}\right).$$

Also if $a>0$

$$\int_a^\infty \left|\sum_2^\infty \frac{(-1)^{n-1}}{\log n} e^{-nixe^{i\delta}}\right|^2 dx$$
$$=\sum_2^\infty \frac{e^{-2na\sin\delta}}{n\log^2 n\sin\delta}+\Sigma\Sigma'\frac{(-1)^{m+n}}{\log m\log n}\frac{e^{ia(me^{-i\delta}-ne^{i\delta})}}{i(me^{-i\delta}-ne^{i\delta})}.$$

The first sum is plainly $O(1/\delta)$, and the second is

$$O\left\{\sum_{m=3}^\infty \sum_{n=2}^{m-1}\frac{e^{-a(m+n)\sin\delta}}{\log m\log n\,(m-n)}\right\}=O\left\{\sum_{m=3}^\infty \frac{e^{-am\sin\delta}}{\log m}\sum_{n=2}^{m-1}\frac{1}{m-n}\right\}$$
$$=O\left\{\sum_{m=3}^\infty e^{-am\sin\delta}\right\},$$

which is also $O(1/\delta)$. This proves (3).

3.44. *Proof of Theorem* 32. Let S be the sub-set of the interval $(T, 2T)$ where $|I|=J$. Then

$$\int_S |I|\,dt=\int_S J\,dt.$$

Now

$$\int_S |I|\,dt \leq \int_T^{2T} |I|\,dt \leq \left\{T\int_T^{2T} |I|^2\,dt\right\}^{\frac{1}{2}} < AH^{\frac{1}{4}}T^{\frac{3}{4}}$$

by 3.41 (2); and, by 3.43 (1) and (2),

$$\int_S J\,dt > T^{-\frac{1}{4}}\int_S (AH+\Psi)\,dt > AT^{-\frac{1}{4}}Hm(S)-T^{-\frac{1}{4}}\int_T^{2T}|\Psi|\,dt$$
$$>AT^{-\frac{1}{4}}Hm(S)-T^{-\frac{1}{4}}\left\{T\int_T^{2T}|\Psi|^2\,dt\right\}^{\frac{1}{2}} > AT^{-\frac{1}{4}}Hm(S)-AT^{\frac{3}{4}},$$

where $m(S)$ is the measure of S. Hence, for H sufficiently large,

$$m(S)<ATH^{-\frac{1}{2}}.$$

Now divide the interval $(T, 2T)$ into $[T/2H]$ pairs of abutting intervals j_1, j_2, each, except the last j_2, of length H, and each j_2 lying immediately to the right of the corresponding j_1. Then either j_1 or j_2 contains a zero of $Z(t)$, unless j_1 consists entirely of points of S. Suppose the latter occurs for ν j_1's. Then

$$\nu H \leq m(S) < ATH^{-\frac{1}{2}}.$$

Hence there are, in $(T, 2T)$, at least

$$[T/2H]-\nu>\frac{T}{H}\left(\frac{1}{3}-\frac{A}{\sqrt{H}}\right)>\frac{T}{4H}$$

zeros. This proves the theorem.

3.51. Theorem 32, interesting as it is, tells us nothing about the general distribution of the zeros; for the total number of zeros is greater than a multiple of $T\log T$. The proportion of zeros which have been proved to lie on $\sigma=\frac{1}{2}$ is therefore infinitesimal.

On the other hand, if we merely ask how many of the zeros lie in the immediate neighbourhood of $\sigma=\frac{1}{2}$, we can say very much more. In fact we can prove, roughly, that all but an infinitesimal proportion of the zeros lie in the immediate neighbourhood of $\sigma=\frac{1}{2}$.

3.52. We begin by proving a general formula concerning the zeros of an analytic function in a rectangle*. Suppose that $\phi(s)$ is meromorphic in and upon the boundary of a rectangle bounded by the lines $t=0$, $t=T$, $\sigma=\alpha$, $\sigma=\beta$ $(\beta>\alpha)$, and regular and not zero on $\sigma=\beta$. The function $\log\phi(s)$ is regular in the neighbourhood of $\sigma=\beta$, and here, starting with any one value of the logarithm, we define $F(s)=\log\phi(s)$. For other points s of the rectangle, we define $F(s)$ to be the value obtained from $\log\phi(\beta+it)$ by continuous variation along $t=$ constant from $\beta+it$ to $\sigma+it$, provided that the path does not cross a zero or pole of $\phi(s)$; if it does, we put

$$F(s)=\tfrac{1}{2}\lim_{\epsilon\to 0}\{F(\sigma+it+i\epsilon)+F(\sigma+it-i\epsilon)\}.$$

Let $\nu(\sigma', T)$ denote the excess of the number of zeros over the number of poles in the part of the rectangle for which $\sigma>\sigma'$, zeros or poles on $t=0$ or $t=T$ counting one-half only. Then

$$\int F(s)\,ds=-2\pi i\int_{\alpha}^{\beta}\nu(\sigma, T)\,d\sigma, \tag{1}$$

the integral on the left being taken round the rectangle in the positive direction.

We may suppose $t=0$ and $t=T$ to be free from zeros and poles of $\phi(s)$; it is easily verified that our halving conventions then ensure the truth of the theorem in the general case.

We have

$$\int F(s)\,ds=\int_{\alpha}^{\beta}F(\sigma)\,d\sigma-\int_{\alpha}^{\beta}F(\sigma+iT)\,d\sigma$$
$$+\int_{0}^{T}\{F(\beta+it)-F(\alpha+it)\}\,i\,dt. \tag{2}$$

* Littlewood (4).

The last term is equal to

$$\int_0^T i\,dt \int_\alpha^\beta \frac{\phi'(\sigma+it)}{\phi(\sigma+it)}\,d\sigma = \int_\alpha^\beta d\sigma \int_\sigma^{\sigma+iT} \frac{\phi'(s)}{\phi(s)}\,ds,$$

and by the theorem of residues

$$\int_\sigma^{\sigma+iT} \frac{\phi'(s)}{\phi(s)}\,ds = \left\{\int_\sigma^\beta + \int_\beta^{\beta+iT} - \int_{\sigma+iT}^{\beta+iT}\right\} \frac{\phi'(s)}{\phi(s)}\,ds - 2\pi i\nu(\sigma, T)$$
$$= F(\sigma+iT) - F(\sigma) - 2\pi i\nu(\sigma, T).$$

Substituting this in (2) we obtain (1).

3.53. THEOREM 33*. *Let $N(\sigma', T)$ denote the number of zeros of $\zeta(s)$ for $\sigma > \sigma'$, $0 < t < T$, and let $-1 \leqq \sigma_0 \leqq 1$. Then*

(1) $$2\pi \int_{\sigma_0}^1 N(\sigma, T)\,d\sigma = \int_0^T \log|\zeta(\sigma_0+it)|\,dt + O(\log T).$$

In 3.52 (1) we take $\alpha=\sigma$, β large and positive, and $\phi(s)=\zeta(s)$, and determine $f(s)=\log\zeta(s)$ as in the case of $F(s)$, starting from the real value at $s=\beta$. Then (on account of the pole at $s=1$)

$$\nu(\sigma, T) = N(\sigma, T) - \tfrac{1}{2} \ (\sigma<1), \quad \nu(\sigma, T) = 0 \ (\sigma \geqq 1).$$

Thus

(2) $$-2\pi i \int_{\sigma_0}^1 \nu(\sigma, T)\,d\sigma = -2\pi i \int_{\sigma_0}^\beta \nu(\sigma, T)\,d\sigma$$
$$= \int_{\sigma_0}^\beta f(\sigma)\,d\sigma - \int_{\sigma_0}^\beta f(\sigma+iT)\,d\sigma - \int_0^T f(\sigma_0+it)\,i\,dt + \int_0^T f(\beta+it)\,i\,dt.$$

Making $\beta\to\infty$, the last term tends to zero, and the first two integrals tend to integrals from σ_0 to ∞. Taking imaginary parts we have

(3) $$2\pi \int_{\sigma_0}^1 \nu(\sigma, T)\,d\sigma = \int_0^T \log|\zeta(\sigma_0+it)|\,dt + \int_{\sigma_0}^\infty \operatorname{am}\zeta(\sigma+iT)\,d\sigma + C(\sigma_0),$$

where $C(\sigma_0)$ is a constant depending on σ_0 only.

Using (20) of the Introduction in the range $(\sigma_0, 2)$, and the fact that $\zeta(s) = 1 + O(e^{-A\sigma})$, $\operatorname{am}\zeta(s) = O(e^{-A\sigma})$, for $\sigma > 2$, we see that the middle term on the right of (3) is $O(\log T)$; and the result follows.

3.54. THEOREM 34†. *For any fixed σ greater than $\tfrac{1}{2}$, $N(\sigma, T) = O(T)$.*

We have, for $\sigma > \tfrac{1}{2}$,

$$\int_2^T \log|\zeta(\sigma+it)|\,dt = \tfrac{1}{2}\int_2^T \log|\zeta(\sigma+it)|^2\,dt$$
$$\leqq \tfrac{1}{2}(T-2)\log\left\{\frac{1}{T-2}\int_2^T |\zeta(\sigma+it)|^2\,dt\right\} \ddagger = O(T),$$

* Littlewood (4). † Bohr and Landau (4), Littlewood (4).

‡ To prove the inequality, replace the integrals by finite sums of which they are the limits, and use the fact that the geometric mean does not exceed the arithmetic mean.

by Theorem 20. Hence, by Theorem 33,

$$\int_{\sigma_0}^{1} N(\sigma, T)\, d\sigma = O(T)$$

for $\sigma_0 > \frac{1}{2}$. Hence, if $\sigma_1 = \frac{1}{2} + \frac{1}{2}(\sigma_0 - \frac{1}{2})$,

$$N(\sigma_0, T) \leqq \frac{1}{\sigma_0 - \sigma_1} \int_{\sigma_1}^{\sigma_0} N(\sigma, T)\, d\sigma \leqq \frac{2}{\sigma_0 - \frac{1}{2}} \int_{\sigma_1}^{1} N(\sigma, T)\, d\sigma = O(T),$$

which is the required result.

From this theorem, and the fact that $N(T) \sim AT \log T$, it follows that *all but an infinitesimal proportion of the zeros of* $\zeta(s)$ *lie in the strip* $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$, *however small* δ *may be.*

3.55. By using Theorem 21 instead of Theorem 20 we can obtain still more precise information about the zeros in the immediate neighbourhood of $\sigma = \frac{1}{2}$. In fact we find at once by the above method that

$$(1) \quad \int_{\sigma_0}^{1} N(\sigma, T)\, d\sigma < AT \log \left[\operatorname{Min}\left\{O(\log T), O\left(\log \frac{1}{\sigma - \frac{1}{2}}\right)\right\}\right],$$

and in particular that

$$(2) \qquad \int_{\frac{1}{2}}^{1} N(\sigma, T)\, d\sigma = O(T \log\log T).$$

We can now put Theorem 34 into a form which holds uniformly with respect to σ.

THEOREM 35*.

$$N(\sigma, T) = O\left(T \frac{1}{\sigma - \frac{1}{2}} \log \frac{1}{\sigma - \frac{1}{2}}\right)$$

uniformly for $\frac{1}{2} < \sigma \leqq 1$.

For as above

$$N(\sigma_0, T) \leqq \frac{2}{\sigma_0 - \frac{1}{2}} \int_{\sigma_1}^{1} N(\sigma, T)\, d\sigma = O\left(\frac{1}{\sigma_0 - \frac{1}{2}}\, T \log \frac{1}{\sigma_1 - \frac{1}{2}}\right),$$

and the result follows.

3.56. THEOREM 36*. *If* $\phi(t)$ *is positive and increases to infinity with* t, *then all but an infinitesimal proportion of the zeros of* $\zeta(s)$ *in the upper half-plane lie in the region*

$$|\sigma - \tfrac{1}{2}| < \phi(t) \log\log t / \log t, \quad t > A.$$

That is to say, the number of zeros outside the region and with imaginary part between 0 and T is $o(T \log T)$.

It is clearly enough to prove that for large T the number of zeros in the region

$$(1) \qquad \sigma \geqq \tfrac{1}{2} + \phi(t) \log\log t / \log t, \quad \sqrt{T} < t < T,$$

* Littlewood (4).

is $o\,(T\log T)$. The curved boundary of (1) lies to the right of $\sigma=\sigma_1$, where
$$\sigma_1-\tfrac{1}{2}=\phi\,(\sqrt{T})\log\log T/\log T.$$
But by Theorem 34
$$N(\sigma_1,T)=O\left(T\frac{1}{\sigma_1-\frac{1}{2}}\log\frac{1}{\sigma_1-\frac{1}{2}}\right)=O\left(T\frac{\log T}{\phi(\sqrt{T})\log\log T}\log\log T\right)$$
$$=O\,\{T\log T/\phi\,(\sqrt{T})\}=o\,(T\log T),$$
and the result follows.

3.61. We now return to the problem of $N(\sigma, T)$ for fixed σ. In this case we can replace the $O\,(T)$ of Theorem 34 by $O\,(T^{\theta})$, where $\theta<1$. We do this by applying the above methods, not to $\zeta\,(s)$ itself, but to the function
$$\phi_z(s)=\zeta(s)\sum_{n<z}\mu(n)\,n^{-s}=\zeta(s)\,\psi_z(s).$$
The zeros of $\zeta\,(s)$ are zeros of $\phi_z(s)$. If $\sigma>1$, $\phi_z(s)\to 1$ as $z\to\infty$. On the Riemann hypothesis this is also true for $\frac{1}{2}<\sigma\leq 1$. Of course we cannot prove this without any hypothesis; but we can choose z so that the additional factor neutralizes to a certain extent the peculiarities of $\zeta\,(s)$, even for values of σ less than 1.

THEOREM 37*. *For any fixed σ greater than $\frac{1}{2}$ and less than* 1
$$N(\sigma, T)=O\,\{T^{1-(2\sigma-1)^2+\epsilon}\}.$$
Since
$$|\psi_z(s)|\geqq 1-\sum_{2\leqq n<z}n^{-2}>0,$$
for $\sigma\geqq 2$, $\phi_z\,(s)$ has no zeros for $\sigma\geqq 2$. Hence, if the $\phi\,(s)$ of § 3.52 is $\phi_z\,(s)$, and $\frac{1}{2}<\sigma_0<1$,
$$-2\pi i\int_{\sigma_0}^{2}\nu(\sigma,T)\,d\sigma=\int_{\sigma_0}^{\infty}F(\sigma)\,d\sigma-\int_{\sigma_0}^{\infty}F(\sigma+iT)\,d\sigma-\int_0^T F(\sigma_0+it)\,i\,dt.$$
Replacing T by $2T$ and subtracting, and taking imaginary parts,
$$2\pi\int_{\sigma_0}^{2}\{\nu(\sigma,2T)-\nu(\sigma,T)\}\,d\sigma$$
$$=\int_T^{2T}\log|\phi_z(\sigma_0+it)|\,dt+\int_{\sigma_0}^{\infty}\{\operatorname{am}\zeta(\sigma+2iT)-\operatorname{am}\zeta(\sigma+iT)$$
$$+\operatorname{am}\psi_z(\sigma+2iT)-\operatorname{am}\psi_z(\sigma+iT)\}\,d\sigma.$$
The integrals involving $\operatorname{am}\zeta\,(s)$ are $O\,(\log T)$, by the concluding remark of § 3.53. Similarly the integrals involving $\operatorname{am}\psi_z\,(s)$ are $O\,(\log z)$; for, as in the proof of (17) of the Introduction, $\operatorname{am}\psi_z\,(s)=O\,(q)$ for $\frac{1}{2}\leqq\sigma\leqq 2$, where q is the number of zeros of
$$\mathbf{R}\,\psi_z(\sigma+it)=\tfrac{1}{2}\{\psi_z(\sigma+it)+\psi_z(\sigma-it)\}$$

* Bohr and Landau (5), Carlson (1), Landau (7).

on $(\frac{1}{2}+it,\ 2+it)$. Since $\psi_z(s)=O(z^A)$, it follows from Jensen's theorem that $q=O(\log z)$. Finally am $\psi_z(s)=O(e^{-A\sigma})$ for $\sigma>2$.

Also, taking any σ $(\frac{1}{2}<\sigma<1)$ and proceeding as in § 3.54,

$$\int_T^{2T}\log|\phi_z(s)|\,dt=\tfrac{1}{2}\int_T^{2T}\log|\phi_z(s)|^2\,dt\leqq\tfrac{1}{2}T\log\left\{\frac{1}{T}\int_T^{2T}|\phi_z(s)|^2\,dt\right\}$$

$$=\tfrac{1}{2}T\log\left\{1+\frac{2}{T}\mathbf{R}\int_T^{2T}\{\phi_z(s)-1\}\,dt+\frac{1}{T}\int_T^{2T}|\phi_z(s)-1|^2\,dt\right\}$$

$$(1)\qquad\leqq\left|\mathbf{R}\int_T^{2T}\{\phi_z(s)-1\}\,dt\right|+\tfrac{1}{2}\int_T^{2T}|\phi_z(s)-1|^2\,dt.$$

We deal with the first term by writing

$$\int_{\sigma+iT}^{\sigma+2iT}\{\phi_z(s)-1\}\,ds=\int_{\sigma+iT}^{2+iT}+\int_{2+iT}^{2+2iT}+\int_{2+2iT}^{\sigma+2iT}.$$

The first and third terms are $O(T^{1-\sigma}z^{1-\sigma})$, since (as is easily seen) $\phi_z(s)$ is of this form. In the second term we write

$$\phi_z(s)-1=\Sigma a_n n^{-s}$$

where $a_1=0$, while, for $n>1$,

$$|a_n|=|\sum_{r|n,\,r<z}\mu(r)|\leqq d(n)=O(n^{\epsilon});$$

and integrating term by term we see that the integral in question is bounded. Hence the first term on the right of (1) is $O(T^{1-\sigma}z^{1-\sigma})$.

3.62. We have now arrived at the kernel of the problem, viz. the consideration of the second term in (1). Here we use Theorem 19 with $x=T$, and obtain

$$\phi_z(s)-1=\sum_{n<T}n^{-s}\sum_{n<z}\mu(n)n^{-s}-1+O(T^{-\sigma})\psi_z(s)$$

$$=\Sigma b_n n^{-s}+O(T^{-\sigma}z^{1-\sigma}),$$

where (if $z<t$) $b_n=0$ for $n<z$ and for $n>zT$; and b_n, like a_n, is $O(n^{\epsilon})$. Hence

$$\int_T^{2T}|\Sigma b_n n^{-s}|^2\,dt=T\Sigma|b_n|^2n^{-2\sigma}+\Sigma\Sigma' b_m b_n(mn)^{-\sigma}\int_T^{2T}(n/m)^{it}\,dt$$

$$=O(T\sum_{n\geqq z}n^{\epsilon-2\sigma})+O\{\sum\sum_{n<m\leqq zT}(mn)^{\epsilon-\sigma}/(\log m/n)\}$$

$$=O(Tz^{1-2\sigma+\epsilon})+O\{(zT)^{2-2\sigma+\epsilon}\}$$

by the lemma of § 2.13. Putting $z=T^{2\sigma-1}$, we obtain

$$\int_T^{2T}|\Sigma b_n n^{-s}|^2\,dt=O(T^{1-(2\sigma-1)^2+\epsilon}).$$

It is easily seen that all the other terms are small compared with this, so that, writing σ_0 for σ in the above result, we now have

$$\int_{\sigma_0}^{2} \{\nu(\sigma, 2T) - \nu(\sigma, T)\}\, d\sigma = O\{T^{1-(2\sigma_0-1)^2+\epsilon}\}.$$

We can replace ν in the integrand (*a fortiori*) by N, and then, replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$, ... and adding,

$$\int_{\sigma_0}^{1} N(\sigma, T)\, d\sigma = O(T^{1-(2\sigma_0-1)^2+\epsilon}).$$

Hence, writing $\lambda = 1/\log T$,

$$N(\sigma_0, T) \leqq \log T \int_{\sigma_0-\lambda}^{\sigma_0} N(\sigma, T)\, d\sigma = O(\log T \,.\, T^{1-(2\sigma_0-2\lambda+1)^2+\epsilon})$$
$$= O(T^{1-(2\sigma_0-1)^2+O(\lambda)+\epsilon}) = O(T^{1-(2\sigma_0-1)^2+\epsilon}).$$

3·63. If we use the approximate functional equation instead of Theorem 19, we obtain a still better result.

THEOREM 38. *For any fixed σ greater than $\frac{1}{2}$*

$$N(\sigma, T) = O\left(T^{1-\frac{2\sigma-1}{3-2\sigma}+\epsilon}\right).$$

We omit the proof*.

3·7. *The remainder in the formula for $N(T)$.*

3·71. We know that

$$(1) \quad N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right)$$

where $\pi S(T) = \operatorname{am} \zeta(\frac{1}{2} + iT)$, the amplitude being found by continuous variation along 2, $2 + iT$, $\frac{1}{2} + iT$. The behaviour of the function $S(T)$ appears to be very complicated. It must have a discontinuity k where T passes through the ordinate of a zero of $\zeta(s)$ of order k (since the term $O(1/T)$ is in fact continuous). Between the zeros $N(T)$ is constant, so that the variation of $S(T)$ must just neutralise that of the other terms.

We know also that $S(T) = O(\log T)$, so that, in the formula as it stands, the term $\frac{7}{8}$ is presumably overwhelmed by the variations of $S(T)$ (though it has not, apparently, been *proved* that $S(T)$ is unbounded). On the other hand if we consider the integrated formula

$$\int_0^T N(t)\, dt = \int_0^T \left(\frac{1}{2\pi} t \log t - \frac{1+\log 2\pi}{2\pi} t + \frac{7}{8}\right) dt + \int_0^T S(t)\, dt + O(\log T)$$

the term in $S(t)$ certainly plays a much smaller part, since, as we shall presently prove, it is still only $O(\log T)$. Presumably this is due to frequent variations in the sign of $S(t)$.

* Titchmarsh (5).

3.72. We write

$$S_1(T) = \int_0^T S(t)\,dt.$$

We shall begin by proving.

THEOREM 39*

$$S_1(T) = \frac{1}{\pi}\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma + iT)|\,d\sigma + O(1).$$

We return to 3.53 (2) with $\sigma_0 = \frac{1}{2}$, $\beta \to \infty$, and this time take the real part. The term in $\nu(\sigma, T)$, being purely imaginary, disappears, and we obtain

$$\int_{\frac{1}{2}}^{\infty} \log|\zeta(\sigma)|\,d\sigma - \int_{\frac{1}{2}}^{\infty} \log|\zeta(\sigma + iT)|\,d\sigma + \int_0^T \pi S(t)\,dt = 0.$$

The first term is constant, and

$$\int_2^{\infty} \log|\zeta(\sigma + iT)|\,d\sigma = \int_2^{\infty} O(e^{-A\sigma})\,d\sigma = O(1).$$

Hence the result.

3.73. THEOREM 40†. $S_1(t) = O(\log t)$.

Integrating 1.83 (1) from s to $2 + it$ (supposing, as we clearly may, that t is not equal to the ordinate of any zero), we obtain

$$\log\zeta(2 + it) - \log\zeta(s) = \sum_{|t-\gamma|<1} \{\log(2 + it - \rho) - \log(s - \rho)\} + O(\log t).$$

Since $\log\zeta(2 + it)$ and $\log(2 + it - \rho)$ are bounded, and there are $O(\log t)$ terms in the sum, this gives

$$\log\zeta(s) = \sum_{|t-\gamma|<1} \log(s - \rho) + O(\log t).$$

Hence

$$\int_{\frac{1}{2}} \log|\zeta(s)|\,d\sigma = \sum_{|t-\gamma|<1} \int_{\frac{1}{2}}^{2} \log|s - \rho|\,d\sigma + O(\log t).$$

The terms of the last sum are bounded, since

$$\tfrac{3}{2}\log(\tfrac{9}{4} + 1) \geqq \int_{\frac{1}{2}}^{2} \log\{(\sigma - \beta)^2 + (\gamma - t)^2\}\,d\sigma \geqq 2\int_{\frac{1}{2}}^{2} \log|\sigma - \beta|\,d\sigma > -A.$$

Hence
$$\int_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma = O(\log t),$$

and the result now follows from Theorem 39.

Another result in the same order of ideas is that

$$\int_0^T \frac{S(t)}{t}\,dt = A + O\left(\frac{\log T}{T}\right) \ddagger.$$

* Cramér (2). † Littlewood (4). ‡ F. and R. Nevanlinna (1), (2).

3.81. THEOREM 41*. *The gaps between the ordinates of successive zeros of $\zeta(s)$ tend to zero.*

We begin by proving that the gaps are bounded. Since it follows from 3.71 (1), and the fact that $S(t) = O(\log t)$, that

$$N(T+H) - N(T) > 0$$

if H is a sufficiently large constant, this really tells us nothing new. But it serves to introduce the method in its simplest form, and the extension is then easily made.

Consider a system of four concentric circles C_1, C_2, C_3, C_4, with centre $2+iT$ and radii $\frac{1}{2}$, 3, 4 and 5 respectively. Suppose that $\zeta(s)$ has no zeros in C_4. Then every branch of $\log \zeta(s)$ is regular in this circle. Let M_1, M_2, M_3 denote the maximum modulus on C_1, C_2, C_3 respectively of the branch obtained in the usual way by continuation along the straight line 2, $2+iT$; and let L be the maximum of its real part on C_4. Then by Carathéodory's theorem

$$M_3 \leqq 8\,\{L + 2\,|\log \zeta(2+iT)|\}.$$

Now $L < A \log T$, since $\zeta(s) = O(t^A)$ in the region considered. Hence $M_3 < A \log T$. Then Hadamard's three-circles theorem gives

$$M_2 \leqq M_1^{\alpha}\, M_3^{\beta},$$

where $$\alpha + \beta = 1, \quad 0 < \beta < 1.$$

Since $M_1 = O(1)$, it follows that

$$M_2 = O(\log^{\beta} T),$$

and in particular that

$$\zeta(-1+iT) = O\{\exp(\log^{\beta} T)\} = O(T^{\epsilon}).$$

But actually it follows from the functional equation that

$$|\zeta(-1+iT)| > AT$$

for all values of T. The above conclusion is therefore untrue, and so every strip $T-5 \leqq t \leqq T+5$ must contain zeros of $\zeta(s)$.

In applying a similar argument to the strip $T-\delta \leqq t \leqq T+\delta$, where δ is arbitrarily small, it is clear that we cannot use four concentric circles. But the ideas of the theorems of Carathéodory and Hadamard are in no way essentially bound up with sets of concentric circles, and our difficulty can be surmounted by using suitable elongated curves instead.

Let D_4 be the rectangle with centre $2+iT$ and a corner at $-3+i(T+\delta)$, the sides being parallel to the axes. We represent D_4 conformally on the unit circle D_4' in the z-plane, so that its centre $2+iT$ corresponds to $z=0$. By this representation a set of concentric circles $|z| = r$ inside

* Littlewood (3).

D_4' will correspond to a set of convex curves inside D_4, such that as $r \to 0$ the curve shrinks upon $2+iT$, while as $r \to 1$ it tends to coincidence with D_4. Let D_1', D_2', D_3' be circles (independent of course of T) for which the corresponding curves D_1, D_2, D_3 in the s-plane pass through the points $\frac{3}{2}+iT$, $-1+iT$, $-2+iT$, respectively.

The proof now proceeds as before. We consider the function

$$f(z) = \log \zeta\{s(z)\},$$

where $s = s(z)$ is the analytic function corresponding to the conformal representation; and we apply the theorems of Carathéodory and Hadamard in the same way as before.

Actually it is possible, by a more detailed study of the functions involved in the conformal representation, to obtain a more precise result. We can prove that $\zeta(s)$ *has a zero* $\beta+i\gamma$ *such that*

$$|\gamma - t| < 16/\log\log\log t$$

for every $t > t_0$.

3.9. *Series of the form* $\Sigma x^\rho f(\rho)$.

3.91. Series of the form $\Sigma x^\rho f(\rho)$, where ρ runs through the complex zeros of $\zeta(s)$, occur in the theory of prime numbers. It is beyond the scope of this work to discuss their application there, but they have interesting connections with the subject of this chapter.

We begin by considering the sum Σx^ρ, where $\rho = \beta + i\gamma$ runs through zeros of $\zeta(s)$ such that $0 < \gamma < T$, and $x > 1$.

THEOREM 42*. *We have*

$$(1) \qquad \sum_{\gamma<T} x^\rho = -\frac{T\log p}{2\pi} + O(\log T)$$

if x is a power of the prime p; otherwise

$$(2) \qquad \sum_{\gamma<T} x^\rho = O(\log T).$$

Let q be a positive number, less than the ordinate of any zero. Then

$$(3) \qquad \left| \int_{2+iq}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} x^s\,ds + i(T-q)\Lambda(x) \right| \leq 2x^2 \sum_1^\infty \frac{\Lambda(n)}{n^2|\log x/n|},$$

$\Lambda(x)$ being zero if x is not an integer. This is easily verified by inserting the series (2″) of the Introduction for $\zeta'(s)/\zeta(s)$, and integrating term by term. Now by the theorem of residues

$$(4) \qquad \int_{2+iq}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} x^s\,ds = \int_{2+iq}^{a+iq} + \int_{a+iq}^{a+iT} + \int_{a+iT}^{2+iT} + 2\pi i \sum_{0<\gamma<T} x^\rho,$$

where $a < 0$, and T is not the ordinate of a zero. It is easily deduced from the functional equation and known properties of the Γ-function that

$$\zeta'(s)/\zeta(s) = O(\log|s|), \quad (\sigma \leq -1,\ t \geq q).$$

* Landau (2).

Hence the second integral on the right tends to zero as $a \to \infty$, and we obtain

$$\int_{2+iq}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, ds = \int_{2+iq}^{-\infty+iq} + \int_{-\infty+iT}^{2+iT} + 2\pi i \sum_{0<\gamma<T} x^{\rho}.$$

Now

$$\int_{-\infty+iT}^{-1+iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, ds = \int_{-\infty}^{-1} O\{\log(|\sigma| + T)\} x^{\sigma} \, d\sigma = O(\log T)$$

and by § 1.83

$$\int_{-1+iT}^{2+iT} \frac{\zeta'(s)}{\zeta(s)} x^s \, ds = \sum_{|T-\gamma|<1} \int_{-1+iT}^{2+iT} \frac{x^s}{s-\rho} \, ds + O(\log T).$$

Writing

$$\int_{-1+iT}^{2+iT} = \int_{-1+iT}^{-1+i(T+2)} + \int_{-1+i(T+2)}^{2+i(T+2)} + \int_{2+i(T+2)}^{2+iT}$$

we see that each term in the sum on the right is $O(1)$. There are $O(\log T)$ terms, so that the sum is $O(\log T)$. The result now follows on combining these equations.

3.92. It was proved by Landau* that the series $\sum \frac{x^{\rho}}{\rho}$ is uniformly convergent in any x-interval which does not include or end at a prime-power. This may be deduced from the previous theorem by partial summation.

3.93. These results also have interesting connections with the formulae of Chapter V, which we anticipate for a moment. On the Riemann hypothesis, the νth ρ is $\frac{1}{2} + i\gamma_{\nu}$, and we have

$$\begin{aligned}\sum_{\gamma<T} x^{\rho} &= \sum_{\nu=1}^{n} x^{\frac{1}{2}+i\gamma_{\nu}} = \sum_{\nu=1}^{n-1} \nu \left(x^{\frac{1}{2}+i\gamma_{\nu}} - x^{\frac{1}{2}+i\gamma_{\nu+1}}\right) + n x^{\frac{1}{2}+i\gamma_n} \\ &= -i \sqrt{x} \log x \sum_{\nu=1}^{n-1} \nu \int_{\gamma_{\nu}}^{\gamma_{\nu+1}} x^{iu} \, du + n x^{\frac{1}{2}+i\gamma_n} \\ &= -i \sqrt{x} \log x \int_{\gamma_1}^{\gamma_n} N(u) x^{iu} \, du + n x^{\frac{1}{2}+i\gamma_n}.\end{aligned}$$

Writing $M(T)$ for the sum of the first three terms on the right of 3.71 (1), we have

$$\begin{aligned} i \log x \int_{\gamma_1}^{\gamma_n} M(u) x^{iu} \, du &= \Big[M(u) x^{iu} \Big]_{\gamma_1}^{\gamma_n} - \int_{\gamma_1}^{\gamma_n} \frac{1}{2\pi} \log \frac{u}{2\pi} x^{iu} \, du \\ &= \{n - S(\gamma_n)\} x^{i\gamma_n} + O(\log \gamma_n) \\ &= n x^{i\gamma_n} + O(\log \gamma_n).\end{aligned}$$

* *Handbuch*, 364–368.

Hence

$$(1)\quad \sum_{\gamma<T} x^{\rho} = -i\sqrt{x}\log x\int_{\gamma_1}^{\gamma_n} S(u)\,x^{iu}\,du + O(\log T),$$

and our result is equivalent to

$$(2)\quad \int_{\gamma_1}^{\gamma_n} S(u)\,x^{iu}\,du = -\frac{i\gamma_n\Lambda(x)}{2\pi\sqrt{x}\log x} + O(\log\gamma_n).$$

Now results such as Theorem 55 suggest that $\pi S(u)$, i.e. am $\zeta(\frac{1}{2}+iu)$, may be connected with sums of the form

$$-\sum_{n<N}\frac{\Lambda_1(n)\sin(u\log n)}{\sqrt{n}}.$$

If we could insert this for $\pi S(u)$ in (2) and integrate term-by-term we should obtain a result of the same type as (2).

It is clear that the numbers ρ are closely connected with the prime numbers. No more explicit relation between them than what is given by the above formulae has been discovered.

CHAPTER IV

THE GENERAL DISTRIBUTION OF VALUES OF $\zeta(s)$

4.1. In the previous chapter we have been concerned almost entirely with the modulus of $\zeta(s)$, and the various values, particularly zero, which it takes. We now consider the problem of $\zeta(s)$ itself, and the values of s for which it takes any given value a.

4.2. One method of dealing with this problem is to connect it with the famous theorem of Picard on functions which do not take certain values. We use the following lemma.*

If $f(s)$ is regular and never 0 or 1 in $|s-s_0|\leqq r$, and $|f(s_0)|\leqq \alpha$, $\theta<1$, then $|f(s)|\leqq A(\alpha,\theta)$ for $|s-s_0|\leqq\theta r$.

From this we deduce

THEOREM 43. *$\zeta(s)$ takes every value, with one possible exception, an infinity of times in $1-\delta<\sigma\leqq 1+\delta$.*

Suppose on the contrary that $\zeta(s)$ takes the distinct values a and b only a finite number of times in the strip, and so *never* above $t=t_0$, say. Let $T>t_0$, and consider the function $f(s)=\{\zeta(s)-a\}/(b-a)$ in the

* See Landau's *Ergebnisse der Funktionentheorie*, § 24, or Valiron's *Integral Functions*, Ch. VI, § 3.

circles C, C' of radii $\frac{1}{2}\delta$ and $\frac{1}{4}\delta$, and common centre $s_0 = 1 + \frac{1}{4}\delta + iT$. Then

$$|f(s_0)| \leqq \alpha = \frac{\zeta(1+\frac{1}{4}\delta) + |a|}{|b-a|},$$

and $f(s)$ is never 0 and 1 in C. Hence, by the lemma,

$$|f(s)| < A(\alpha)$$

in C', and so $|\zeta(\sigma + iT)| < A(a, b, \alpha)$ for $1 \leqq \sigma \leqq 1 + \frac{1}{8}\delta$, $T > t_0$. Hence $\zeta(s) = O(1)$ in $\sigma > 1$, which is false, by Theorem 1. This proves the theorem.

We should, of course, expect the exceptional value to be 0.

4.3. If we assume the Riemann hypothesis, we can use a similar method inside the critical strip; but more detailed results independent of the Riemann hypothesis can be obtained by the method of Diophantine approximation. We devote the rest of the chapter to developments of this method.

4.31. We restrict ourselves in the first place to the half-plane $\sigma > 1$*; and we consider, not $\zeta(s)$ itself, but $\log \zeta(s)$, viz. the function defined for $\sigma > 1$ by the series

$$\log \zeta(s) = -\sum_p (p^{-s} + \tfrac{1}{2}p^{-2s} + \dots).$$

We consider at the same time the function

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \log p\,(p^{-s} + p^{-2s} + \dots).$$

We observe that both the functions are represented by Dirichlet series, absolutely convergent for $\sigma > 1$, and capable of being written in the form

$$F(s) = f_1(p_1^{-s}) + f_2(p_2^{-s}) + \dots$$

where $f_n(x)$ is a power series in x whose coefficients do not depend on s. In what follows $F(s)$ denotes either of the two functions.

4.32. We first consider the values which $F(s)$ takes on the line $\sigma = \sigma_0$, where σ_0 is an arbitrary number greater than 1. On this line

$$F(s) = \sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0} e^{-it \log p_n}),$$

and, as t varies, the amplitudes $-t \log p_n$ are of course all related. But we shall see that there is an intimate connection between the set U of values assumed by $F(s)$ on $\sigma = \sigma_0$ and the set V of values assumed by the function

$$\Phi(\sigma_0, \phi_1, \phi_2 \dots) = \sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0} e^{2i\pi\phi_n})$$

of an infinite number of *independent* real variables $\phi_1, \phi_2, \dots$.

* Bohr (3), (4).

We shall in fact shew that *the set* U, *which is obviously contained in* V, *is everywhere dense in* V, *i.e. that corresponding to every value* v *in* V (*i.e. to every given set of values* $\phi_1, \phi_2, \ldots$), *and every* ϵ, *there exists a* t *such that*

$$|F(\sigma_0 + it) - v| < \epsilon.$$

Since the Dirichlet series from which we start is absolutely convergent for $\sigma = \sigma_0$, it is obvious that we can find $N = N(\sigma_0, \epsilon)$ such that

$$(1) \qquad \left| \sum_{n=N+1}^{\infty} f_n(p_n^{-\sigma_0} e^{2i\pi\mu_n}) \right| < \tfrac{1}{3}\epsilon$$

for any values of the μ_n, and in particular for $\mu_n = \phi_n$, or for

$$\mu_n = -\frac{t \log p_n}{2\pi}.$$

Now since the numbers $\log p_n$ are linearly independent, we can, by Kronecker's theorem, find a number t and integers $g_1, g_2, \ldots g_N$ such that

$$(2) \qquad |-t \log p_n - 2\pi\phi_n - 2\pi g_n| < \eta \quad (n = 1, 2, \ldots N),$$

η being an assigned positive number. Since $f_n(p_n^{-\sigma_0} e^{2i\pi\phi})$ is, for each n, a continuous function of ϕ, we can suppose η so small that

$$(3) \qquad \left| \sum_{n=1}^{N} \{f_n(p_n^{-\sigma_0} e^{2i\pi\phi_n}) - f_n(p_n^{-\sigma_0} e^{-it \log p_n})\} \right| < \tfrac{1}{3}\epsilon.$$

The result now follows from (1) and (3).

4.33. We next consider the set W of values which $F(s)$ takes 'in the immediate neighbourhood' of the line $\sigma = \sigma_0$, i.e. the set of all values of w such that the equation $F(s) = w$ has, for every positive δ, a root in the strip $|\sigma - \sigma_0| < \delta$.

In the first place, it is evident that U is contained in W. Further it is easy to see that U is everywhere dense in W. For, for sufficiently small δ (e.g. for $\delta < \frac{1}{2}(\sigma_0 - 1)$),

$$|F'(s)| < A(\sigma_0)$$

for all values of s in the strip $|\sigma - \sigma_0| < \delta$, so that

$$(1) \qquad |F(\sigma_0 + it) - F(\sigma_1 + it)| < A(\sigma_0)|\sigma_1 - \sigma_0| \qquad (|\sigma_1 - \sigma_0| < \delta).$$

Now each w in W is assumed by $F(s)$ either on the line $\sigma = \sigma_0$, in which case it is a u, or at points $\sigma_1 + it$ arbitrarily near the line, in which case, in virtue of (1), we can find a u such that $|w - u| < A(\sigma_0)|\sigma_1 - \sigma_0| < \epsilon$.

We now proceed to prove that W *is identical with* V. Since U is contained in and is everywhere dense in both V and W, it follows that

each of V and W is everywhere dense in the other. It is therefore obvious that W is contained in V, if V is closed.

We shall see presently that much more than this is true, viz. that V consists of all the points of an area, including the boundary. The following direct proof that V is closed, however, is very instructive.

Let $\bar{v}$ be a limit-point of V, and let v_ν $(\nu = 1, 2, \ldots)$ be a sequence of v's tending to $\bar{v}$. To each v_ν corresponds a point P_ν, $(\phi_1^{(\nu)}, \phi_2^{(\nu)}, \ldots)$, in the space of an infinite number of dimensions defined by $0 \leqq \phi_n^{(\nu)} < 1$, $(n = 1, 2, \ldots)$, such that $\Phi(\sigma_0, \phi_1^{(\nu)}, \ldots) = v_\nu$.

Now since (P_ν) is a bounded set of points (i.e. all the coordinates are bounded), it has* a limit-point $\bar{P}$, $(\bar{\phi}_1, \bar{\phi}_2, \ldots)$, i.e. a point such that from (P_ν) we can choose a sequence (P_{ν_r}) such that each coordinate $\phi_n^{(\nu_r)}$ of P_{ν_r} tends to the limit $\bar{\phi}_n$ as $r \to \infty$.

It is now easy to prove that $\bar{P}$ corresponds to $\bar{v}$, i.e. that $\Phi(\sigma_0, \bar{\phi}_1, \ldots) = \bar{v}$, so that $\bar{v}$ is a point of V. For the series for v_{ν_r}, viz.

$$\sum_{n=1}^{\infty} f_n\left(p_n^{-\sigma_0} e^{2i\pi\phi_n^{(\nu_r)}}\right),$$

is uniformly convergent with respect to r, since (by Weierstrass' M-test) † it is uniformly convergent with respect to all the ϕ's; further, the nth term tends to $f_n(p_n^{-\sigma_0} e^{2i\pi\bar{\phi}_n})$ as $r \to \infty$. Hence

$$\bar{v} = \lim_{r\to\infty} v_{\nu_r} = \lim_{r\to\infty} \sum_{n=1}^{\infty} f_n\left(p_n^{-\sigma_0} e^{2i\pi\phi_n^{(\nu_r)}}\right) = \Phi(\sigma_0, \bar{\phi}_1, \ldots),$$

which proves our result.

4.34. To establish the identity of V and W it remains to prove that V is contained in W. It is obviously sufficient (and also necessary) for this that W should be closed. But that W is closed does not follow, as might perhaps be supposed, from the mere fact that W is the set of values taken by a bounded analytic function in the immediate neighbourhood of a line. Thus e^{-z^2} is bounded and arbitrarily near to zero in every strip including the real axis, but never actually assumes the value zero. The fact that W is closed (which we shall not prove directly) depends on the special nature of the function $F(s)$.

Let $v = \Phi(\sigma_0, \phi_1, \phi_2, \ldots)$ be an arbitrary value contained in V. We have to shew that v is a member of W, i.e. that, in every strip $|\sigma - \sigma_0| < \delta$, $F(s)$ assumes the value v. We consider, together with $F(s)$, the function

$$F_1(s) = \sum_{n=1}^{\infty} f_n\left(p_n^{-s} e^{2i\pi\phi_n}\right),$$

* We use the extension to an infinity of variables of a well-known theorem. See Whittaker and Watson, *Modern Analysis* (ed. 4, 1927), § 2.21.

† *Ibid.* § 3.34.

which is obviously regular for $\sigma > 1$, and not constant. At $s = \sigma_0$, $F_1(s)$ takes the value v. We choose a small circle C_1, with centre σ_0, such that $F_1(s) \neq v$ on the circumference. Let C be this circle translated along $\sigma = \sigma_0$ through a distance t_0. We shall first shew, what is the main point of the proof, that Kronecker's theorem enables us to choose t_0 such that, for every pair of corresponding points s in C and s_1 in C_1, the difference $|F(s) - F_1(s_1)|$ is less than an arbitrarily given ϵ.

The argument is almost exactly the same as that used to shew that U is everywhere dense in V. The series for $F(s)$ and $F_1(s)$ are uniformly convergent in the strip, and, for each fixed N, $\sum_{n=1}^{N} f_n(p_n^{-\sigma} e^{2i\pi\mu_n})$ is a continuous function of $\sigma, \mu_1, \dots \mu_N$. It is therefore sufficient to shew that we can choose t_0 so that the difference between the amplitudes of p_n^{-s} at $s = \sigma_0 + it_0$, and $p_n^{-s} e^{2i\pi\phi_n}$ at $s = \sigma_0$ respectively, and consequently that between the respective amplitudes at every pair of corresponding points of the two circles, is (*modulo* 2π) arbitrarily small for $n = 1, 2, \dots N$. The possibility of the choice follows at once from Kronecker's theorem.

4.35. It is now easy to see, by means of a well-known result* in the theory of functions, that, when the ϵ in the inequality $|F(s) - F_1(s_1)| < \epsilon$ is sufficiently small, $F(s)$ must assume the value v inside the circle C. For let m be the minimum value of $|F_1(s_1) - v|$ on C_1, so that $m > 0$, and let us choose $\epsilon = m$. Then we have, for values of s_1 on C_1,

$$|F(s_1 + it_0) - F_1(s_1)| < m.$$

Now
$$F(s_1 + it_0) - v = G(s_1) + H(s_1),$$

where
$$G(s_1) = F_1(s_1) - v, \quad H(s_1) = F(s_1 + it_0) - F_1(s_1).$$

Then $|H(s_1)| < |G(s_1)|$ on C_1, and hence $G(s_1) + H(s_1)$ has the same number of zeros inside C_1 as $G(s_1)$; and so, since $G(\sigma_0) = 0$, it has at least one such zero. Consequently $F(s)$ takes the value v at some point inside C. This completes the proof that V is contained in W, and therefore the proof that V and W are identical.

4.36. We now proceed to the study of the set V. Let V_n be the set of values taken by $f_n(p_n^{-s})$ for $\sigma = \sigma_0$, i.e. the set taken by $f_n(x)$ for $|x| = p_n^{-\sigma_0}$. Then V is the 'sum' of the sets of points $V_1, V_2, \dots$, i.e. it is the set of all values $v_1 + v_2 + \dots$, where v_1 is any point of V_1, v_2 any point of V_2, and so on. For the function $\log \zeta(s)$, V_n consists of the points of the curve described by $-\log(1-x)$ as x describes the circle $|x| = p_n^{-\sigma_0}$; for $\zeta'(s)/\zeta(s)$ it consists of the points of the curve described by $-(x \log p_n)/(1-x)$.

* Goursat's *Cours d'Analyse* (ed. 3, 1918), t. 2, § 307.

4.41. We begin by considering the function $\zeta'(s)/\zeta(s)$. In this case we can find the set V explicitly. The curve of values of V_n is a circle with its centre at

$$c_n = -p_n^{-2\sigma_0}\log p_n/(1-p_n^{-2\sigma_0})$$

and with radius

$$\rho_n = p_n^{-\sigma_0}\log p_n/(1-p_n^{-2\sigma_0}).$$

Let $c=\Sigma c_n = \zeta'(2\sigma_0)/\zeta(2\sigma_0)$. Then V is the set of all the values of

$$c+\Sigma\,\rho_n e^{i\theta_n}$$

for independent $\theta_1, \theta_2, \ldots$.

The set V' of the values $\Sigma\,\rho_n e^{i\theta_n}$ is the sum of an infinite number of circles with zero as common centre, whose radii $\rho_1, \rho_2, \ldots$ form, as it is easy to see, a decreasing sequence. Now for such a sum of circles we have the following theorem*:

(i) *If* $\rho_1 > \rho_2+\rho_3+\ldots$, *then the set* V' *consists of all points* z *of the annulus*

$$\sum_{n=1}^{\infty}\rho_n \geqq |z| \geqq \rho_1 - \sum_{n=2}^{\infty}\rho_n.$$

(ii) *If* $\rho_1 \leqq \rho_2+\rho_3+\ldots$, *then* V' *consists of all points* z *for which*

$$|z| \leqq \sum_{n=1}^{\infty}\rho_n.$$

In the case before us we have an explicit expression for ρ_n; it is therefore easy to see when we are in cases (i) and (ii) respectively.

The complete result† is that there is an absolute constant $D=2{\cdot}57\ldots$, determined as the root of the equation

$$\frac{2^{-D}\log 2}{1-2^{-2D}} = \sum_{n=2}^{\infty}\frac{p_n^{-D}\log p_n}{1-p_n^{-2D}},$$

such that for $\sigma_0 > D$ we are in case (i) and for $1<\sigma_0\leqq D$ we are in case (ii). The greater radius in each case is

$$R=\zeta'(2\sigma_0)/\zeta(2\sigma_0)-\zeta'(\sigma_0)/\zeta(\sigma_0);$$

the lesser radius in case (i) is

$$r=2\rho_1 - R = 2^{1-\sigma_0}\log 2/(1-2^{-2\sigma_0}) - R.$$

4.42. Summing up, we have the following results for $\zeta'(s)/\zeta(s)$.

* The main ideas involved in the proof of this may be acquired by constructing the various figures corresponding to the case of three circles. For the details of the investigation see Bohr (4) and (6).

† Burrau (1).

THEOREM 44. *The values which $\zeta'(s)/\zeta(s)$ takes on the line $\sigma=\sigma_0$ form a set everywhere dense in a region $R(\sigma_0)$. If $\sigma_0 > D$, $R(\sigma_0)$ is the annulus (boundary included) with centre c and radii R and r; if $\sigma_0 \leqq D$, $R(\sigma_0)$ is the circular area (boundary included) with centre c and radius R; c, R, r are continuous functions of σ_0 defined by*

$$c=\zeta'(2\sigma_0)/\zeta(2\sigma_0), \quad R=c-\zeta'(\sigma_0)/\zeta(\sigma_0), \quad r=2^{1-\sigma_0}\log 2/(1-2^{-2\sigma_0})-R.$$

Further, as $\sigma_0 \to \infty$,

$$\lim c=\lim R=\lim r=0, \quad \lim c/R=\lim (R-r)/R=0;$$

as $\sigma_0 \to D$, $\lim r=0$; and as $\sigma_0 \to 1$, $\lim R=\infty$, $\lim c=\zeta'(2)/\zeta(2)$.

THEOREM 45. *The set of values which $\zeta'(s)/\zeta(s)$ takes in the immediate neighbourhood of $\sigma=\sigma_0$ is identical with $R(\sigma_0)$. In particular, since c tends to a finite limit, and R to infinity, as $\sigma_0 \to 1$, $\zeta'(s)/\zeta(s)$ takes all values infinitely often in the strip $1<\sigma<1+\delta$, for an arbitrary positive δ.*

The above results evidently enable us to study the set of points $\{\sigma_0(a)\}$, where $\sigma=\sigma_0(a)>1$ is a line on which $\zeta'(s)/\zeta(s)$ takes the assigned value a. We confine ourselves to giving the result for $a=0$; this is the most interesting case, since the zeros of $\zeta'(s)/\zeta(s)$ are identical with those of $\zeta'(s)$.

THEOREM 46. *There is an absolute constant E, between 2 and 3, such that the set $\{\sigma_0(0)\}$ is everywhere dense between 1 and E, and has no member greater than E. That is, $\zeta'(s) \neq 0$ for $\sigma > E$, while $\zeta'(s)$ has an infinity of zeros in every strip between $\sigma=1$ and $\sigma=E$.*

4.51. We proceed now to the study of $\log \zeta(s)$. We have seen that in this case the set of points V_n consists of all the points of the curve C_n described by $-\log(1-x)$ as x describes the circle $|x|=p_n^{-\sigma_0}$; the set of points V is therefore the 'sum' of the curves C_n.

The difficulty in this case is that C_n is not a circle. But it is a closed convex curve including the origin. Now the essential property of a circle which is bound up with the theorem on 'summation' with which we have been concerned is precisely the quality of convexity. Accordingly the theory of summation of circles can be extended to apply to convex curves, in the following form:

The set of points determined by the sum of an 'absolutely convergent' series of convex curves is either (i) *the region bounded by two convex curves, of which one is entirely interior to the other, or* (ii) *the region bounded by a single convex curve. In each case the boundary is included as part of the region.*

Further, let the curves C_1, C_2, ... *be arranged in order of decreasing area, and let* ρ_n' *and* ρ_n'' *be the maximum and minimum radii vectores from the origin to* C_n. *Finally let us suppose that the origin is interior to all the curves* C_n. *Then we are in case* (i) *if* $\rho_1' > \sum_{n=2}^{\infty} \rho_n'$, *and in case* (ii) *if* $\rho_1' \leqq \sum_{n=2}^{\infty} \rho_n''$. *In any case the origin is interior to the outer boundary* Γ *of the region, and (what is obvious), if* R', R'' *are the maximum and minimum distances from the origin to* Γ, *we have*

$$R' \leqq \Sigma \rho_n', \quad R'' \geqq \Sigma \rho_n''.$$

If $f(z)$ is an analytic function regular in $|z| \leqq r$, a sufficient condition that $f(z)$ should describe a convex curve as z describes $|z| = r$ is that the tangent to the path of $f(z)$ should rotate steadily through 2π as z describes the circle; i.e. it is sufficient that am $\{zf'(z)\}$ should increase steadily through 2π. This condition is satisfied in the case $f(z) = -\log(1-z)$; for $zf'(z) = z/(1-z)$ describes a circle including the origin as z describes $|z| = r < 1$.

It is easy to see further that the convex curve described by $-\log(1-z)$ includes the origin, and increases in area as r increases. Now

$$A\,|z| > |\log(1-z)| > A\,|z| \quad (|z| \leqq \tfrac{1}{2}),$$

and therefore, for the curve C_n,

$$\rho_n' < A p_n^{-\sigma_0}, \quad \rho_n'' > A p_n^{-\sigma_0}.$$

This shews at once that when σ_0 is sufficiently large we are in case (i), and that when σ_0 is sufficiently near 1 we are in case (ii), the region $R(\sigma_0)$ containing an arbitrarily large circle with its centre at the origin.

It is only in the case of circles that the criteria given in the theorem enable us to determine in all circumstances whether we are in case (i) or case (ii); for arbitrary convex curves the criteria deal only with the extreme cases. They do not, for example, settle the question whether there exists an absolute constant D' such that we are in cases (i) or (ii) according as $\sigma_0 > D'$ or $1 < \sigma_0 \leqq D'$. The discussion of this point demands a closer investigation of the geometry of the special curves with which we are dealing, and the question would appear to be one of considerable intricacy.

4.52. The relations between U, V and W give us the following analogues for $\log \zeta(s)$ of the results for $\zeta'(s)/\zeta(s)$.

THEOREM 47. *On each line* $\sigma = \sigma_0 > 1$ *the values of* $\log \zeta(s)$ *are everywhere dense in a region* $R(\sigma_0)$, *which is either* (i) *the ring-shaped area bounded by two convex curves, or* (ii) *the area bounded by one convex curve. For*

sufficiently large values of σ_0 *we are in case* (ii), *and for values of* σ_0 *sufficiently near to* 1 *we are in case* (i).

THEOREM 48. *The set of values which* $\log \zeta(s)$ *takes in the immediate neighbourhood of* $\sigma = \sigma_0$ *is identical with* $R(\sigma_0)$. *In particular, since* $R(\sigma_0)$ *includes any given finite area when* σ_0 *is sufficiently near* 1, $\log \zeta(s)$ *takes every value an infinity of times in* $1 < \sigma < 1 + \delta$.

As a consequence of the last result, we have:

THEOREM 49. *The function* $\zeta(s)$ *takes every value except* 0 *infinitely often in the strip* $1 < \sigma < 1 + \delta$.

This is a more precise form of Theorem 43.

4.61. We have seen above that $\log \zeta(s)$ takes any assigned value a an infinity of times in $\sigma > 1$. It is natural to raise the question *how often* the value a is taken, i.e. the question of the behaviour for large T of the number $M_a(T)$ of roots of $\log \zeta(s) = a$ in $\sigma > 1$, $0 < t < T$. This question is evidently closely related to the question as to *how often*, as $t \to \infty$, the point $(a_1 t, a_2 t, \dots a_N t)$ of Kronecker's theorem, which, in virtue of the theorem, comes (*modulo* 1) arbitrarily near every point in the N-dimensional unit cube, comes within a given distance of an assigned point $(b_1, b_2, \dots b_N)$. The answer to this last question is given by the following theorem, which asserts that, roughly speaking, the point $(a_1 t, \dots a_N t)$ comes near every point of the unit cube equally often, i.e. it does not give a preference to any particular region of the unit cube.

Let $a_1, \dots a_N$ *be linearly independent, and let* γ *be a region of the* N*-dimensional unit cube with volume* V. *Let* $I_\gamma(T)$ *be the sum of the intervals between* $t = 0$ *and* $t = T$ *for which the point* $P(a_1 t, \dots a_N t)$ *is* (*modulo* 1) *inside* γ. *Then*

$$\lim_{T\to\infty} I_\gamma(T)/T = V.$$

If we call a point with coordinates of the form $(a_1 t, \dots a_N t)$, *modulo* 1, an 'accessible' point, Kronecker's theorem states that the accessible points are everywhere dense in the unit cube C. If now γ_1, γ_2 are two cubes with sides parallel to the axes, and with centres at accessible points P_1 and P_2, corresponding to $t = t_1$ and t_2, it is easily seen that

$$\lim I_{\gamma_1}(T)/I_{\gamma_2}(T) = 1.$$

For $(a_1 t, \dots a_N t)$ will lie inside γ_2 when and only when $\{a_1(t + t_2 - t_1), \dots\}$ lies inside γ_1. Consider now a set of p non-overlapping cubes c, inside C, of side ϵ, each of which has its centre at an accessible point, and q of which lie inside γ; secondly a set of overlapping similar cubes c' such that C is included in P of them and γ in Q of them. Since the accessible points are everywhere dense, it is possible to choose the cubes such that

q/P and Q/p are arbitrarily near to V. Now denoting by $\sum_\gamma I_c(T)$ the sum of t-intervals in $(0, T)$ corresponding to the cubes c which lie in γ, and so on,

$$\sum_\gamma I_c(T)/\sum_C I_{c'}(T) \leqq \frac{I_\gamma(T)}{T} \leqq \sum_\gamma I_{c'}(T)/\sum_C I_c(T).$$

Making $T \to \infty$ we obtain

$$\frac{q}{P} \leqq \overline{\lim_{T\to\infty}} \frac{I_\gamma(T)}{T} \leqq \frac{Q}{p},$$

and the result follows.

4.62. We can now prove:

THEOREM 50. *There are positive constants $A(a)$ and $A'(a)$ such that the number $M_a(T)$ of zeros of $\log\zeta(s)-a$ in $\sigma>1$ satisfies the inequalities*

$$A(a)T < M_a(T) < A'(a)T.$$

We deduce the lower bound from the more general result that *if $\sigma=\sigma_0$ is a line on which $\log\zeta(s)$ comes arbitrarily near to the given number a, then in every strip $\sigma_0-\delta<\sigma<\sigma_0+\delta$ the value a is taken more than $A(a, \sigma_0, \delta)T$ times, for large T, in $0<t<T$.*

To prove this we have to consider our former argument of § 4.36, used to establish the existence of a root of $\log\zeta(s)=a$ in the strip, and use Kronecker's theorem in its generalized form. We saw that a sufficient condition that $\log\zeta(s)=a$ may have a root inside a circle with centre σ_0+it_0 and radius 2δ is that, for a certain N and corresponding numbers $\phi_1, \ldots \phi_N$, and a certain $\eta=\eta(\sigma_0, \delta, \phi_1, \ldots \phi_N)$,

$$|-t_0\log p_n - 2\pi\phi_n - 2\pi g_n| < \eta \quad (n=1, 2, \ldots N).$$

From the generalized Kronecker's theorem it follows that the sum of the intervals, between 0 and T, in which t_0 satisfies this condition, is asymptotically equal to $(\eta/2\pi)^N T$, and it is therefore greater than $\frac{1}{2}(\eta/2\pi)^N T$ for large T. Hence we can select more than $\frac{1}{8}(\eta/2\pi)^N T/\delta$ numbers t_0' in them, no two of which differ by less than 4δ. If now we describe circles with the points σ_0+it_0' as centres and radius 2δ, these circles will not overlap, and each of them will contain a zero of $\log\zeta(s)-a$. This gives the desired result.

As to the upper bound for $M_a(T)$, it follows from the more general result that *if b is any given constant, the number of zeros of $\zeta(s)-b$ in $\sigma>\frac{1}{2}+\delta$, $0<t<T$ is $O(T)$ as $T\to\infty$.*

The proof of this is substantially the same as that of Theorem 34, the function $\zeta(s)-b$ playing the same part that $\zeta(s)$ did there. Finally the number of zeros of $\log\zeta(s)-a$ is not greater than the number of zeros of $\zeta(s)-e^a$, and so is $O(T)$.

4.7. We now turn to the more difficult question of the behaviour of $\zeta(s)$ in the critical strip*. The difficulty of course is that $\zeta(s$ is no longer represented by an absolutely convergent Dirichlet series. But, by a device like that used in the proof of Theorem 37, we are able to obtain, in the critical strip, results analogous to those already obtained in the region of absolute convergence. For $\sigma \leqq 1$, $\log \zeta(s)$ is defined, on each line $t = \text{constant}$ which does not pass through a singularity, by continuation along this line from $\sigma > 1$.

THEOREM 51. *Let σ_0 be a fixed number in the range $\frac{1}{2} < \sigma \leqq 1$. Then the values which* $\log \zeta(s)$ *takes on $\sigma = \sigma_0$, $t > 0$, are everywhere dense in the whole plane.*

Let
$$\zeta_N(s) = \zeta(s) \prod_{n=1}^{N} (1 - p_n^{-s}).$$
This function is very similar to the $\phi_z(s)$ of Theorem 37, but it happens to be more convenient here.

Let δ be a positive number less than $\frac{1}{2}(\sigma_0 - \frac{1}{2})$. Then it is easily seen from the proof of Theorem 37 that, for $N \geqq N_0(\sigma_0, \epsilon)$, $T \geqq T_0 = T_0(N)$,
$$\int_1^T |\zeta_N(\sigma + it) - 1|^2 \, dt < \epsilon T$$
uniformly for $\sigma_0 - \delta \leqq \sigma \leqq \sigma_1 + \delta \quad (\sigma_1 > 1)$. Hence
$$\int_1^T \int_{\sigma_0 - \delta}^{\sigma_1 + \delta} |\zeta_N(\sigma + it) - 1|^2 \, d\sigma \, dt < (\sigma_1 - \sigma_0 + 2\delta) \, \epsilon T.$$
Hence
$$\int_{\nu - \frac{1}{2}}^{\nu + \frac{1}{2}} \int_{\sigma_0 - \delta}^{\sigma_1 + \delta} |\zeta_N(\sigma + it) - 1|^2 \, d\sigma \, dt < \sqrt{\epsilon} \, (\sigma_1 - \sigma_0 + 2\delta)$$
for more than $(1 - \sqrt{\epsilon})T$ values of ν. For such values of ν, and $\nu - \frac{1}{2} + \delta \leqq t \leqq \nu + \frac{1}{2} - \delta$, $\sigma_0 \leqq \sigma \leqq \sigma_1$, Cauchy's integral gives
$$|\zeta_N(s) - 1|^2 = \left| \frac{1}{2\pi} \int_0^{2\pi} \{\zeta_N(s + re^{i\theta}) - 1\}^2 \, d\theta \right|$$
for $0 < r < \delta$; and so, multiplying by r and integrating,
$$\left|\zeta_N(s) - 1\right|^2 = \left| \frac{1}{\pi\delta^2} \int_0^{2\pi} \int_0^{\delta} \{\zeta_N(s + re^{i\theta}) - 1\}^2 \, r \, dr \, d\theta \right| < \frac{\sqrt{\epsilon} \, (\sigma_1 - \sigma_0 + 2\delta)}{\pi\delta^2},$$
since the circle of integration is included in one of the previous rectangles of integration. Hence, *having given $\eta < 1$, $\eta' < 1$, we can choose δ and ϵ so that*
$$(1) \qquad |\zeta_N(\sigma + it) - 1| < \eta \quad (\sigma_0 \leqq \sigma \leqq \sigma_1)$$
for a set of values of t of measure greater than $(1 - \eta')T$, and for
$$N \geqq N_0(\sigma_0, \eta, \eta'), \quad T \geqq T_0(N).$$

Let
$$R_N(s) = -\sum_{N+1}^{\infty} \operatorname{Log}(1 - p_n^{-s}) \quad (\sigma > 1),$$

* Bohr (7), (8), Bohr and Courant (1).

where Log denotes the principal value. Then $\zeta_N(s) = \exp\{R_N(s)\}$. We want to shew that $R_N(s) = \operatorname{Log} \zeta_N(s)$, i.e. that $|\mathbf{I} R_N(s)| < \frac{1}{2}\pi$, for $\sigma \geqq \sigma_0$ and the values of t for which (i) holds. This is true for $\sigma = \sigma_1$, if σ_1 is sufficiently large, since $|R_N(s)| \to 0$ as $\sigma \to \infty$. Also, by (1), $\mathbf{R}\, \zeta_N(s) > 0$ for $\sigma_0 \leqq \sigma \leqq \sigma_1$, so that $\mathbf{I} R_N(s)$ must remain between $\pm \frac{1}{2}\pi$ for all values of σ in this interval. This gives the desired result.

We have therefore

$$|R_N(s)| = |\operatorname{Log}[1 + \{\zeta_N(s) - 1\}]| < 2\,|\zeta^N(s) - 1| < 2\eta$$

for $\sigma = \sigma_0$, $N \geqq N_0(\sigma_0, \eta, \eta')$, $T \geqq T_0(N)$, *in a set of values of t greater than* $(1 - \eta')\,T$.

We now consider the function

$$F_N(\sigma_0 + it) = -\sum_{n=1}^{N} \log(1 - p_n^{-(\sigma_0 + it)}) = -\sum_{n=1}^{N} \log(1 - r_n e^{-it \log p_n}),$$

where $r_n = p_n^{-\sigma_0}$; and in conjunction with it the function of n independent variables

$$\Phi_N(\phi_1, \ldots \phi_N) = -\sum_{n=1}^{N} \log(1 - r_n e^{2\pi i \phi_n}).$$

The set V of values of Φ_N is determined by the theorem on convex curves. The maximum and minimum radii vectores from 0 to C_n are

$$\rho_n' = -\log(1 - r_n), \quad \rho_n'' = \log(1 + r_n).$$

Since $\Sigma \rho_n''$ is divergent, it is easily seen that we are in case (ii) of the theorem if N is large enough, and that, for sufficiently large values of N, V includes any finite region of the complex plane. In particular, if a is any given number, we can find a number N and values of the ϕ's such that $\Phi_N(\phi_1, \ldots \phi_N) = a$.

We can then, by Kronecker's theorem, find a number t such that $|F_N(\sigma_0 + it) - a|$ is arbitrarily small. But this in itself is not sufficient to prove the theorem, since this value of t does not necessarily make $|R_N(s)|$ small. An additional argument is therefore required.

$$\text{Let } \Phi_{M,N} = -\sum_{n=M+1}^{N} \log(1 - r_n e^{2\pi i \phi_n}) = \sum_{n=M+1}^{N} \sum_{m=1}^{\infty} \frac{r_n^{\,m}}{m} e^{2\pi i m \phi_n}.$$

Then, expressing the squared modulus of this as the product of conjugates, and integrating term-by-term, we obtain

$$\int_0^1 \int_0^1 \cdots \int_0^1 |\Phi_{M,N}|^2\, d\phi_{M+1} \ldots d\phi_N = \sum_{n=M+1}^{N} \sum_{m=1}^{\infty} \frac{r_n^{\,2m}}{m^2}$$

$$< \sum_{n=M+1}^{N} r_n^{\,2} \sum_{m=1}^{\infty} m^{-2} < A \sum_{M+1}^{\infty} r_n^{\,2},$$

which can be made arbitrarily small, by choice of M, for all N. It therefore follows from the theory of Riemann integration of a continuous function that, having given ϵ, we can divide up the $(N-M)$-dimensional unit cube into the sub-cubes q_ν, each of volume λ, in such a way that

$$\lambda \sum_{\nu} \operatorname*{Max}_{q_\nu} |\Phi_{M,N}|^2 < \tfrac{1}{2}\epsilon^2.$$

Hence *for* $M \geqq M_0(\epsilon)$, *and any* $N > M$, *we can find cubes of total volume greater than* $\frac{1}{2}$ *in which* $|\Phi_{M,N}| < \epsilon$.

We now choose our value of t as follows:

(i) Choose M so large, and give $\phi_1', \dots \phi_M'$ such values, that

$$-\sum_{n=1}^{M} \log(1 - r_n e^{2\pi i\phi_n'}) = a.$$

It then follows from considerations of continuity that, having given ϵ, we can find an M-dimensional cube with centre $(\phi_1', \dots \phi_M')$ and side $d > 0$ throughout which

$$\left| -\sum_{n=1}^{M} \log(1 - r_n e^{2\pi i\phi_n}) - a \right| < \tfrac{1}{3}\epsilon.$$

(ii) We may also suppose that M has been chosen so large that, for any value of N, $|\Phi_{M,N}| < \frac{1}{3}\epsilon$ in certain $(N-M)$-dimensional cubes of total volume greater than $\frac{1}{2}$.

(iii) Having fixed M and d, we choose N so large that, for $T > T_0(N)$, the inequality $|R_N(s)| < \frac{1}{3}\epsilon$ holds in a set of values of t of measure greater than $(1 - \frac{1}{2}d^M)\,T$.

(iv) Let $I(T)$ be the sum of the intervals between 0 and T for which the point

$$(-t\log p_1/2\pi, \dots -t\log p_N/2\pi)$$

is, *modulo* 1, inside one of the N-dimensional cubes, of total volume greater than $\frac{1}{2}d^M$, determined by the above construction. Then by the extended Kronecker's theorem, $I(T) > \frac{1}{2}d^M T$ if T is large enough. There are therefore values of t for which the point lies in one of these cubes, and for which at the same time $|R_N(s)| < \frac{1}{3}\epsilon$. But for such a value of t

$$|\log\zeta(s) - a| \leqq |F_N(s) - a| + |R_N(s)| < \tfrac{2}{3}\epsilon + \tfrac{1}{3}\epsilon = \epsilon,$$

and the result follows.

4.81. THEOREM 52. *Let* $\frac{1}{2} < \alpha < \beta < 1$, *and let* a *be any complex number. Let* $L_{a,\alpha,\beta}(T)$ *be the number of zeros of* $\log\zeta(s) - a$ *(defined as before) in the rectangle* $\alpha < \sigma < \beta$, $0 < t < T$. *Then there are positive constants* $A(a, \alpha, \beta)$, $A'(a, \alpha, \beta)$ *such that*

$$A(a, \alpha, \beta)\,T < L_{a,\alpha,\beta}(T) < A'(a, \alpha, \beta)\,T \quad (T > T_0).$$

This is the extension of Theorem 50 which, in view of Theorem 51, we should now expect. The proof is too long to be given here*. But in the proofs of Theorems 50 and 51 the reader has already acquired the main ideas on which it depends.

4.82. An immediate corollary of Theorem 52 is that *if* $M_{b,\alpha,\beta}(T)$ *is the number of points in the strip* $\alpha < \sigma < \beta$, $0 < t < T$, *where* $\zeta(s) = b$ $(b \neq 0)$, *then* $M_{b,\alpha,\beta}(T) > A(b, \alpha, \beta)\, T$ *for* $T > T_0$.

This, in conjunction with Theorem 37, shews that the value 0 of $\zeta(s)$, if it occurs at all in $\sigma > \frac{1}{2}$, is at any rate quite exceptional, zeros being infinitely rarer than b-values for any value of b other than zero.

CHAPTER V

CONSEQUENCES OF THE RIEMANN HYPOTHESIS

5.11. This chapter is on an entirely different footing from the previous ones. We assume throughout the truth of the unproved Riemann hypothesis, that all the complex zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$. So all the theorems in this chapter may be untrue.

There are two reasons for investigations of this kind. If the Riemann hypothesis is true, it will presumably be proved some day. These theorems will then take their place as an essential part of the theory. If it is false, we may perhaps hope in this way sooner or later to arrive at a contradiction. Actually the theory, as far as it goes, is perfectly coherent, and shews no signs of breaking down.

The Riemann hypothesis, of course, leaves nothing more to be said about the 'horizontal' distribution of the zeros. But from it we can deduce interesting consequences both about the 'vertical' distribution of the zeros, and about our 'order' problems. In all cases we obtain much more precise results with the hypothesis than without it. But even a proof of the Riemann hypothesis would not by any means complete the theory. The finer shades in the behaviour of $\zeta(s)$ would still not be completely determined.

* That $\zeta(s)$ takes every value except zero infinitely often in the critical strip was proved by Bohr and Landau (3), on the assumption that the Riemann hypothesis is true. On the Riemann hypothesis the proof is much shorter.

5.12. THEOREM 53*. *We have*

(1) $$\log \zeta(s) = O\{(\log t)^{2-2\sigma+\epsilon}\}$$

uniformly for $\frac{1}{2} < \sigma_0 \leqq \sigma \leqq 1.$

Here, and in most of the chapter, we use the Riemann hypothesis in the form '$\log \zeta(s)$ is regular for $\sigma > \frac{1}{2}$ (except at $s = 1$).'

We first prove that

(2) $$\log \zeta(s) = O(\log t)$$

uniformly for $\sigma \geqq \frac{1}{2} + \delta\,(\delta > 0)$. We apply Carathéodory's theorem to the function $\log \zeta(s)$ and the circles with centre $2 + it$ and radii $\frac{3}{2} - \frac{1}{2}\delta$ and $\frac{3}{2} - \delta$. On the larger circle

$$\mathbf{R}\{\log \zeta(s)\} = \log |\zeta(s)| < A \log t.$$

Also $\log \zeta(2 + it) = O(1)$. Hence on the smaller circle

$$\operatorname{Max} |\log \zeta(s)| < \frac{2(\frac{3}{2} - \frac{1}{2}\delta)}{\frac{1}{2}\delta}\{O(\log t) + O(1)\},$$

and the result follows.

We now apply Hadamard's three-circles theorem to the circles with centre $\sigma_1 + it\,(\sigma_1 > 1)$ and radii

$$r_1 = \sigma_1 - 1 - \delta, \quad r_2 = \sigma_1 - \sigma, \quad r_3 = \sigma_1 - \tfrac{1}{2} - \delta,$$

where $\frac{1}{2} + \delta < \sigma_0 \leqq \sigma$. The function $\log \zeta(s)$ is bounded on the smallest circle, and satisfies (2) on the largest. Let M be its maximum modulus on the middle circle. Then

$$M < A(\delta)(\log t)^{\log \frac{r_2}{r_1} / \log \frac{r_3}{r_1}}.$$

Now

$$\log \frac{r_2}{r_1} \Big/ \log \frac{r_3}{r_1} = \log\left(1 + \frac{1 + \delta - \sigma}{\sigma_1 - 1 - \delta}\right) \Big/ \log\left(1 + \frac{\frac{1}{2}}{\sigma_1 - 1 - \delta}\right) < 2(1 + \delta - \sigma) + \epsilon$$

if $\sigma_1 = \sigma_1(\delta, \epsilon)$ is sufficiently large. Hence

$$\log \zeta(s) = O\{(\log t)^{2(1+\delta-\sigma)+\epsilon}\}.$$

Since both δ and ϵ may be arbitrarily small, the result follows.

5.13. Since the index of $\log t$ in the above result is less than unity if ϵ is small enough, it follows that

$$-\epsilon \log t < \log |\zeta(s)| < \epsilon \log t \quad (t > t_0(\epsilon)),$$

i.e. we have both

(1), (2) $$\zeta(s) = O(t^{\epsilon}), \quad \frac{1}{\zeta(s)} = O(t^{\epsilon}),$$

* Littlewood (1).

for $\sigma > \frac{1}{2}$. In particular, Lindelöf's function $\mu(\sigma)$ is zero for $\sigma > \frac{1}{2}$, and so also, by convexity, for $\sigma = \frac{1}{2}$. Thus on the Riemann hypothesis it is completely determined for all values of σ.

5.21. THEOREM 54*. *The series* $\Sigma\mu(n)n^{-s}$ *is convergent, and its sum is* $1/\zeta(s)$, *for all values of* σ *greater than* $\frac{1}{2}$.

Let

$$M(x) = \sum_{n \leq x} \mu(n). \tag{1}$$

Consider the integral

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{x^s}{s}\frac{1}{\zeta(s)}\,ds = \sum_{n=1}^{\infty} \mu(n)\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s}.$$

We may suppose that x is half an odd integer. If $n < x$, the theorem of residues gives

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s} = 1 - \frac{1}{2\pi i}\left[\int_{-\infty-iT}^{2-iT} + \int_{2+iT}^{-\infty+iT}\right].$$

Now

$$\int_{-\infty+iT}^{2+iT}\left(\frac{x}{n}\right)^s\frac{ds}{s} = O\left\{\frac{1}{T}\int_{-\infty}^{2}\left(\frac{x}{n}\right)^\sigma d\sigma\right\} = O\left\{\frac{(x/n)^2}{T\log x/n}\right\} = O\left(\frac{x^3}{Tn^2}\right),$$

since $\log x/n > A/x$. A similar result holds for the other integral. If $n > x$, we take the integrals along $(2+iT, \infty+iT)$ and $(2-iT, \infty-iT)$, and there is no residue term. Hence

$$M(x) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{s}\frac{1}{\zeta(s)}\,ds + O\left(\frac{x^3}{T}\right).$$

Since the integrand is regular for $\sigma > \frac{1}{2}$, we may replace the integral by integrals along the straight lines joining $2-iT$, $\frac{1}{2}+\delta-iT$, $\frac{1}{2}+\delta+iT$, $2+iT$.

Using 5.13 (2) we see that the integral along $\sigma = \frac{1}{2}+\delta$ is

$$O\left\{x^{\frac{1}{2}+\delta}\int_{-T}^{T}(1+|t|^{-1-\epsilon})\,dt\right\} = O(x^{\frac{1}{2}+\delta}T^\epsilon),$$

while the remaining integrals are

$$O\left\{\int_{\frac{1}{2}+\delta}^{2} x^\sigma T^{-1+\epsilon}\,d\sigma\right\} = O(x^2T^{-1+\epsilon}).$$

Taking $T = x^3$ we obtain

$$M(x) = O(x^{\frac{1}{2}+\delta+3\epsilon}) + O(1),$$

or simply

$$M(x) = O(x^{\frac{1}{2}+\epsilon}). \tag{2}$$

* Littlewood (1).

It now follows by partial summation that the series $\Sigma\mu(n)n^{-s}$ is uniformly convergent for $\sigma \geqq \sigma_0 > \frac{1}{2}$; and since it represents $1/\zeta(s)$ for $\sigma > 1$, it represents it for $\frac{1}{2} < \sigma \leqq 1$ also, by the theory of analytic continuation.

The convergence of $\Sigma\mu(n)n^{-s}$ for $\sigma > \frac{1}{2}$ is therefore a necessary and sufficient condition for the truth of the Riemann hypothesis.

5.22. The result 5.21 (2) can be replaced by the more precise one*

$$(1) \qquad M(x) = O\left\{x^{\frac{1}{2}}\exp\left(A\,\frac{\log x}{\log\log x}\right)\right\}.$$

This, however, requires the much more difficult analysis sketched in § 5.71. No more in this direction is known. This is in contrast to the corresponding results in the theory of primes, in which $\zeta'(s)/\zeta(s)$ occurs in the same way that $1/\zeta(s)$ occurs here, and in which it is only a power of $\log x$ that is in doubt. Problems involving $1/\zeta(s)$ appear to be exceptionally difficult.

The hypothesis that

$$(2) \qquad M(x) = O(x^{\frac{1}{2}})$$

is attributed to Mertens. It is supported by a certain amount of numerical evidence†, though there is nothing in the general theory to suggest it. Indeed the analogy between $M(x)$ and the remainder in the formula for $\pi(x)$ suggests that $M(x)$ may really contain a slowly increasing factor, such as $\log\log\log x$, which would not affect it appreciably as far as the calculations go.

If Mertens' hypothesis is true, all the zeros of $\zeta(s)$ are simple ‡.

For if $\sigma > \frac{1}{2}$,

$$(3) \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} M(n)\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} = s\int_1^{\infty}\frac{M(u)}{u^{s+1}}\,du,$$

and if (2) is true it follows that

$$\frac{1}{\zeta(s)} = O\left(s\int_1^{\infty}\frac{du}{u^{\sigma+\frac{1}{2}}}\right) = O\left(\frac{s}{\sigma-\frac{1}{2}}\right).$$

Hence every zero on $\sigma = \frac{1}{2}$ must be simple; and also, if $\frac{1}{2}+i\gamma$ is a zero,

$$(4) \qquad \frac{1}{\zeta'(\frac{1}{2}+i\gamma)} = \lim_{\sigma\to\frac{1}{2}}\frac{\sigma-\frac{1}{2}}{\zeta(\sigma+i\gamma)} = O(\gamma).$$

There is no great difficulty in deducing from this an inequality for $1/\zeta'(\frac{1}{2}+it)$ for all values of t.

* Landau (8), Titchmarsh (3). † v. Sterneck (1).

‡ Cramér and Landau (1).

In view of the suspicion under which (2) rests, it is of some interest to notice that, in order to prove that all the zeros are simple, it would be sufficient to know that

(5) $$M(x) = o(x^{\frac{1}{2}} \log x);$$

for then we deduce from (3) that, for any fixed t, as $\sigma \to \frac{1}{2}$,

$$\frac{1}{\zeta(s)} = o\left\{\int_1^\infty \frac{\log u}{u^{\sigma+\frac{1}{2}}}\, du\right\} = o\left\{\frac{1}{(\sigma-\frac{1}{2})^2}\right\},$$

and this would be false if t were the ordinate of a double zero.

In any case no more than (2) can be true, that is to say

(6) $$M(x) = \Omega(x^{\frac{1}{2}}).$$

This is true without any hypothesis. For if the result is false it follows as in § 5.21 that the Riemann hypothesis is true. So if the Riemann hypothesis is false, the result is true. We have therefore only to consider the case in which the Riemann hypothesis is true. But then, if (6) is untrue, i.e. if $M(x) = o(\sqrt{x})$, we deduce from (3) that, for fixed t, as $\sigma \to \frac{1}{2}$,

$$\frac{1}{\zeta(s)} = o\left\{\int_1^\infty \frac{du}{u^{\sigma+\frac{1}{2}}}\right\} = o\left\{\frac{1}{\sigma-\frac{1}{2}}\right\}.$$

This is false if t is the ordinate of a zero, so that (6) is proved.

5.3. *The function** $\nu(\sigma)$.

5.31. We now define a function $\nu(\sigma)$, for $\sigma > \frac{1}{2}$, as the order of $\log \zeta(\sigma+it)$ in $\log t$; that is to say as the lower bound of numbers ξ such that $\log \zeta(s) = O(\log^{\xi} t)$.

It follows from the absolute convergence of the Dirichlet series for $\log \zeta(s)$ that $\nu(\sigma) \leqq 0$ for $\sigma > 1$. It then follows (as in § 5.12) from Carathéodory's theorem, and the fact that $\zeta(s) = O(t^A)$, that $\nu(\sigma) < \infty$ for $\sigma > \frac{1}{2}$ (actually that $\nu(\sigma) \leqq 1$).

Further, $\nu(\sigma)$ is a convex function of σ. This is proved by the method of Phragmén and Lindelöf, as in the case of the function† $\mu(\sigma)$. We consider

$$g(s) = \log \zeta(s)\,\{\log(-is)\}^{k(s)-\epsilon},$$

where $k(s)$ is the linear function of s which is equal to $\nu(\sigma_1)$ for $\sigma = \sigma_1$ and to $\nu(\sigma_2)$ for $\sigma = \sigma_2$; $g(s)$ is bounded for $\sigma = \sigma_1$, $\sigma = \sigma_2$, and so uniformly in the strip, and the result follows from this.

Hence also $\nu(\sigma)$, like $\mu(\sigma)$, is continuous and non-decreasing in any range where it is finite. Now the Dirichlet series for $\log \zeta(s)$ shews

* Bohr and Landau (3), Littlewood (5). † See § 1.52.

that $\nu(\sigma)=0$ for $\sigma>\sigma_1$, where $2^{-\sigma_1}=\sum_{n=3}^{\infty}\Lambda_1(n)\,n^{-\sigma_1}$; then a slight adaptation of the argument which proves convexity shews that $\nu(\sigma)$ cannot be negative (and in particular cannot be $-\infty$) for $\sigma>\frac{1}{2}$.

Hence $\nu(\sigma)=0$ for $\sigma>1$, and $0\leqq\nu(\sigma)\leqq 1$, and $\nu(\sigma)$ is convex, for $\sigma>\frac{1}{2}$. Incidentally this gives a new proof of Theorem 53.

5.32. The exact value of $\nu(\sigma)$ is not known for any value of σ less than 1. All we know is

THEOREM 55. *For* $\frac{1}{2}<\sigma<1$,

$$1-\sigma\leqq\nu(\sigma)\leqq 2(1-\sigma).$$

The upper bound is Theorem 53, and the lower bound follows at once from Theorem 17. The same lower bound can however be obtained in another and in some respects simpler way, though this proof, unlike the former, depends essentially upon the Riemann hypothesis*. Before we come to the proof, however, we require some new formulae involving $\zeta(s)$.

5.41. *The approximate functional equation for $\zeta'(s)/\zeta(s)$.*

In § 2.21 we obtained the approximate functional equation for $\zeta(s)$ by combining the formulae Σn^{-s} and $\chi\Sigma n^{s-1}$ which represent the function for $\sigma>1$ and $\sigma<0$ respectively. In the case of $\zeta'(s)/\zeta(s)$ we have also two formulae, this time both valid for $\sigma>1$:

$$(1)\qquad -\frac{\zeta'(s)}{\zeta(s)}=\Sigma\frac{\Lambda(n)}{n^s}=-b+\frac{1}{s-1}+\frac{1}{2}\frac{\Gamma'(1+\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}s)}-\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

We can combine these in the formula†

$$(2)\qquad -\frac{\zeta'(s)}{\zeta(s)}=\sum_{n\leqq x}{}'\frac{\Lambda(n)}{n^s}-\frac{x^{1-s}}{1-s}-\sum_{q=1}^{\infty}\frac{x^{-2q-s}}{2q+s}+\sum_{\rho}\frac{x^{\rho-s}}{\rho-s}$$

which holds for $x>1$, $s\neq 1$, $s\neq -2q$, $s\neq\rho$.

The last three terms in this formula can, on the Riemann hypothesis, be replaced by comparatively small error terms, so that we obtain an approximate functional equation for $\zeta'(s)/\zeta(s)$, valid for $\sigma>\frac{1}{2}$.

This, however, is not the formula we actually use. We have already (in §§ 2.42, 2.51) had some experience of the advantage of using series of the form $\Sigma a_n e^{-\delta n}$, where δ is small, as an approximation to Σa_n, rather than the ordinary partial sum $\sum_{n<N} a_n$. Approximations of the former type are much easier to prove, and are just as effective in applications.

* Bohr and Landau (3), Littlewood (5). † *Handbuch*, 353 (9).

5.42. THEOREM 56*. *As* $t \to \infty$,

$$(1)\quad \frac{-\zeta'(s)}{\zeta(s)} = \sum \frac{\Lambda(n)}{n^s} e^{-\delta n} + \sum_{\rho} \delta^{s-\rho}\, \Gamma(\rho - s) + O(\delta^{\sigma-\frac{1}{4}} \log t)$$

uniformly for $\frac{1}{2} \leqq \sigma \leqq \frac{9}{8}$, $e^{-\surd t} \leqq \delta \leqq 1$.

We have (as in § 2.42)

$$(2)\quad \sum \frac{\Lambda(n)}{n^s} e^{-\delta n} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w-s) \frac{\zeta'(w)}{\zeta(w)} \delta^{s-w}\, dw.$$

We move the contour to $R(w) = \frac{1}{4}$, that is, we apply Cauchy's theorem to the rectangle $\frac{1}{4} \pm iT$, $2 \pm iT$, and make $T \to \infty$ through certain values which avoid the poles of the integrand. Since the Γ-function tends exponentially to zero, there is no particular difficulty in justifying this process, e.g. by using Theorem 18. The integrand has poles at $w = s$, with residue $\zeta'(s)/\zeta(s)$; at the points $w = \rho$, with residues $\Gamma(\rho - s)\, \delta^{s-\rho}$ (we count multiple zeros in the sum with their proper order of multiplicity); and at $w = 1$, with residue

$$-\Gamma(1-s)\, \delta^{s-1} = O(e^{-At}) = O(\delta^{\sigma-\frac{1}{4}} \log t).$$

It remains only to verify that

$$\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \Gamma(w-s) \frac{\zeta'(w)}{\zeta(w)} \delta^{s-w}\, dw = O(\delta^{\sigma-\frac{1}{4}} \log t).$$

Now, by the lemma of § 1.83 and the Riemann hypothesis,

$$\zeta'(w)/\zeta(w) = O(\log|2+v|),$$

if $w = \frac{1}{4} + iv$; and

$$\Gamma(w-s) = O(e^{-A|v-t|}), \quad |\delta^{s-w}| = \delta^{\sigma-\frac{1}{4}}.$$

The result now follows, since

$$\int_{-\infty}^{\infty} e^{-A|v-t|} \log|v+2|\, dv = O(\log t) \int_{0}^{2t} e^{-A|v-t|}\, dv$$
$$+ O\left\{\int_{-\infty}^{\infty} e^{-\frac{1}{2}A|v|} \log|v+2|\, dv\right\}$$
$$= O(\log t) + O(1) = O(\log t).$$

5.43. The above formula enables us to put Theorem 53 into a more precise form.

THEOREM 57*. *We have*

$$(1)\quad \frac{\zeta'(s)}{\zeta(s)} = O\{(\log t)^{2-2\sigma}\}, \quad \log \zeta(s) = O\left\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\right\},$$

uniformly for $\quad \frac{1}{2} < \sigma_0 \leqq \sigma \leqq \sigma_1 < 1.$

* Littlewood (5).

By Theorem 56

$$\frac{\zeta'(s)}{\zeta(s)}=O\left\{\Sigma\frac{\Lambda(n)}{n^{\sigma}}e^{-\delta n}\right\}+O\{\delta^{\sigma-\frac{1}{2}}\sum_{\rho}|\Gamma(\rho-s)|\}+O(\delta^{\sigma-\frac{1}{2}}\log t).$$

Now

$$(2)\quad \Sigma\frac{\Lambda(n)}{n^{\sigma}}e^{-\delta n}$$

$$=-\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(w-s)\frac{\zeta'(w)}{\zeta(w)}\delta^{\sigma-w}\,dw\sim\Gamma(1-\sigma)\,\delta^{\sigma-1},$$

the integral being dominated by the pole at $w=1$. Also, by the asymptotic formula for $\Gamma(z)$,

$$|\Gamma(\rho-s)|<Ae^{-A|\gamma-t|},$$

uniformly for σ in the above range. Hence

$$\sum_{\rho}|\Gamma(\rho-s)|<A\sum_{\gamma}e^{-A|t-\gamma|}=A\sum_{n=1}^{\infty}\sum_{n-1\leqq|t-\gamma|<n}e^{-A|t-\gamma|},$$

and, since there are less than $A\log(t+n+2)$ terms in the inner sum, this is less than

$$A\sum_{n=1}^{\infty}\log(t+n+2)\,e^{-An}<A\log(2t+2)\sum_{n<t}e^{-An}+A\sum_{n>t}\log(2n+2)\,e^{-An}$$

$$=O(\log t).$$

Hence

$$\frac{\zeta'(s)}{\zeta(s)}=O(\delta^{\sigma-1})+O(\delta^{\sigma-\frac{1}{2}}\log t)+O(\delta^{\sigma-\frac{1}{2}}\log t),$$

and taking $\delta=(\log t)^{-2}$ we obtain the first result.

Again, for $\sigma_0\leqq\sigma\leqq\sigma_1$,

$$\log\zeta(s)=\log\zeta(\sigma_1+it)-\int_{\sigma}^{\sigma_1}\frac{\zeta'(u+it)}{\zeta(u+it)}\,du$$

$$=O\{(\log t)^{2-2\sigma_1+\epsilon}\}+O\left\{\int_{\sigma}^{\sigma_1}(\log t)^{2-2u}\,du\right\}$$

$$=O\{(\log t)^{2-2\sigma_1+\epsilon}\}+O\{(\log t)^{2-2\sigma}/\log\log t\}.$$

If $\sigma\leqq\sigma_2<\sigma_1$ and $\epsilon<2(\sigma_1-\sigma_2)$, this is of the required form; and since σ_1 and so σ_2 may be as near to 1 as we please, the result (with σ_2 for σ_1) follows.

5.44. For the purpose of Theorem 55 we require an approximate formula for $\log\zeta(s)$.

THEOREM 58*. *For fixed a and σ such that $\frac{1}{2}<a<\sigma\leqq 1$, and for $e^{-\sqrt{t}}\leqq\delta\leqq 1$,*

$$(1)\quad \log\zeta(s)=\Sigma\Lambda_1(n)\,n^{-s}e^{-\delta n}+O\{\delta^{\sigma-a}(\log t)^{\nu(a)+\epsilon}\}+O(1).$$

The proof begins in the same way as that of Theorem 56, but this

* Littlewood (5).

time we move the contour as far as $\mathbf{R}(w)=\alpha$ only, and the sum in ρ does not occur. If $w=\alpha+iv$,

$$\frac{\zeta'(w)}{\zeta(w)}=\frac{1}{2\pi i}\int_{|z-w|=\epsilon}\frac{\log\zeta(z)}{(z-w)^2}dz=O\left[\frac{1}{\epsilon}\{\log(|v|+2)\}^{\nu(\alpha-\epsilon)+\epsilon}\right],$$

or, since $\nu(\sigma)$ is continuous, simply

$$\zeta'(w)/\zeta(w)=O\left[\{\log(|v|+2)\}^{\nu(\alpha)+\epsilon}\right].$$

Hence

$$\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)\frac{\zeta'(w)}{\zeta(w)}\delta^{s-w}\,dw$$
$$=O\left[\delta^{\sigma-\alpha}\int_{-\infty}^{\infty}e^{-A|v-t|}\{\log(|v|+2)\}^{\nu(\alpha)+\epsilon}\,dv\right],$$

and, as in § 5.42, the last integral is $O\{(\log t)^{\nu(\alpha)+\epsilon}\}$. Hence

$$-\frac{\zeta'(s)}{\zeta(s)}=\Sigma\frac{\Lambda(n)}{n^s}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+o(1).$$

This result holds uniformly in the range $(\sigma, \frac{9}{8})$, and so we may integrate over this interval. We obtain

$$\log\zeta(s)-\Sigma\Lambda_1(n)\,n^{-s}\,e^{-\delta n}-O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}-o(1)$$
$$=\log\zeta(\tfrac{9}{8}+it)-\Sigma\Lambda_1(n)\,n^{-\frac{9}{8}-it}\,e^{-\delta n}=O(1).$$

5.45. *New proof that** $\nu(\sigma)\geqq 1-\sigma$. Theorem 58 enables us to extend the method of Diophantine approximation, already used for $\sigma>1$, to values of σ between $\frac{1}{2}$ and 1. We have, by Theorem 58,

$$(1)\quad \mathbf{R}\log\zeta(s)$$
$$=\Sigma\Lambda_1(n)\,n^{-\sigma}\cos(t\log n)\,e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1)$$
$$=\sum_{n\leqq N}\Lambda_1(n)\,n^{-\sigma}\cos(t\log n)\,e^{-\delta n}+O\left(\sum_{n>N}e^{-\delta n}\right)$$
$$+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1)$$

for all values of N. Now by Dirichlet's theorem (with $\tau=2\pi$) there is a number t, $2\pi\leqq t\leqq 2\pi q^N$, and integers $x_1, \dots x_N$, such that, for given N and q,

$$|t\log n/2\pi-x_n|\leqq 1/q\quad(n=1, 2, \dots N).$$

Let us assume for the moment that this number t satisfies the condition of Theorem 58, that $e^{-\sqrt{t}}\leqq\delta$. For this t,

$$\sum_{n\leqq N}\Lambda_1(n)\,n^{-\sigma}\cos(t\log n)e^{-\delta n}>\sum_{n\leqq N}\Lambda_1(n)\,n^{-\sigma}\{1+O(1/q)\}\,e^{-\delta n}$$
$$=\sum_{n\leqq N}\Lambda_1(n)\,n^{-\sigma}\,e^{-\delta n}+O(1/q)\sum_{n\leqq N}n^{-\sigma}.$$

* Littlewood (5).

The last term is $O(q^{-1} N^{1-\sigma})$, and

$$\sum_{n \leqq N} \Lambda_1(n) n^{-\sigma} e^{-\delta n} \geqq \frac{1}{\log N} \sum_{n \leqq N} \Lambda(n) n^{-\sigma} e^{-\delta n} \geqq \frac{1}{\log N} \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} e^{-\delta n}$$
$$+ O\left(\sum_{N}^{\infty} e^{-\delta n}\right) > \frac{A(\sigma)\delta^{\sigma-1}}{\log N} + O\left(\frac{e^{-\delta N}}{\delta}\right),$$

by § 5.43 (2). We therefore obtain, from (1),

$$(2) \quad \mathbf{R} \log \zeta(s) > A(\sigma) \frac{\delta^{\sigma-1}}{\log N} + O\left(\frac{e^{-\delta N}}{\delta}\right) + O\left(\frac{N^{1-\sigma}}{q}\right)$$
$$+ O\{\delta^{\sigma-a} (\log t)^{\nu(a)+\epsilon}\} + O(1).$$

Let us now take $q = N = [\delta^{-a}]$, where $a > 1$. Then the second and third terms on the right are $O(1)$, and, since

$$\log t \leqq A + N \log q < A \delta^{-a} \log(1/\delta),$$

we obtain

$$(3) \qquad \mathbf{R} \log \zeta(s) > (\log t)^{1-\sigma+\eta} + O\{(\log t)^{a-\sigma+\nu(a)+\eta'}\},$$

where η and η' are functions of ϵ and a which tend to zero with ϵ and $a-1$.

If the first term on the right of (3) is of larger order than the second, it follows at once that $\nu(\sigma) \geqq 1-\sigma$. Otherwise

$$a - \sigma + \nu(a) \geqq 1 - \sigma.$$

Since the second alternative reduces to the first when $a \to \sigma$, the result follows.

5.46. We have still to shew that the t of the above argument satisfies $e^{-\sqrt{t}} \leqq \delta$. Suppose that, on the contrary, $\delta < e^{-\sqrt{t}}$ (for some arbitrarily small values of δ). Now, as in §1.13, for $\sigma > 1$,

$$(1) \quad \mathbf{R}\, \zeta(\sigma + it) > \left(\cos \frac{2\pi}{q} - \frac{2}{N^{\sigma-1}}\right) \frac{1}{\sigma - 1} > \left(\frac{1}{2} - \frac{2}{N^{\sigma-1}}\right) \frac{1}{\sigma - 1}$$

for $q \geqq 6$. Taking $\sigma = 1 + \log 8/\log N$,

$$(2) \qquad \mathbf{R}\, \zeta(\sigma + it) > \frac{1}{4(\sigma-1)} = \frac{\log N}{4 \log 8} > A \log \frac{1}{\delta} > A \sqrt{t}.$$

But since $\mathbf{R}\, \zeta(\sigma + it) \to \infty$, by (1), and $t > 2\pi$, it follows that $t \to \infty$, and (2) contradicts Theorem 6. This completes the proof.

5.5. *The line* $\sigma = 1$. On the Riemann hypothesis the order of $\zeta(s)$ on $\sigma = 1$ is known, and the only remaining problems are those of the exact values of the constants.

5.51. THEOREM 59*. *We have*

(1) $$|\log \zeta(1+it)| \leqq \log\log\log t + A.$$

In particular

(2) $$\zeta(1+it) = O(\log\log t), \quad 1/\zeta(1+it) = O(\log\log t).$$

Taking $\sigma = 1$, $\alpha = \frac{3}{4}$, in Theorem 58, we have

$$|\log \zeta(1+it)| < \Sigma \Lambda_1(n) n^{-1} e^{-\delta n} + O(\delta^{\frac{1}{4}} \log t) + O(1).$$

Now

$$\Sigma \Lambda_1(n) n^{-1} e^{-\delta n} = \sum_{n<\delta^{-2}} \Lambda_1(n) n^{-1} e^{-\delta n} + O\Big(\sum_{n \geqq \delta^{-2}} e^{-\delta n}\Big)$$

$$< \sum_{p^m < \delta^{-2}} p^{-m} + O(1)$$

$$< \sum_{p<\delta^{-2}} p^{-1} + \sum_p (p^2 - p)^{-1} + O(1).$$

Here the second sum is $O(1)$, and†

$$\sum_{p<x} p^{-1} = \log\log x + O(1),$$

so that

$$\Sigma \Lambda_1(n) n^{-1} e^{-\delta n} = \log\log 1/\delta + O(1),$$

$$|\log \zeta(1+it)| < \log\log 1/\delta + O(\delta^{\frac{1}{4}} \log t) + O(1).$$

Taking $\delta = (\log t)^{-4}$ the result follows.

5.52. Comparing 5.51 (2) with Theorems 7 and 10, we see that, as far as the order of the functions is concerned, the result is final. It is however possible by an elaboration of the analysis to obtain interesting results about the constants. We know already, without any hypothesis, that

(1) $$\overline{\lim} \frac{\zeta(1+it)}{\log\log t} \geqq e^{\gamma},$$

where γ is Euler's constant. On the Riemann hypothesis we can prove that

(2) $$\overline{\lim} \frac{\zeta(1+it)}{\log\log t} \leqq 2\beta(1) e^{\gamma},$$

where

$$\beta(1) = \lim_{\sigma \to 1-0} \tfrac{1}{2}\nu(\sigma)/(1-\sigma).$$

It follows from Theorem 55 that $\beta(1) \leqq 1$, so that (1) and (2) differ by a factor 2 at most. If, as is quite possible, $\nu(\sigma) = 1 - \sigma$, then $\beta(1) = \frac{1}{2}$, and the constant is determined exactly.

* Littlewood (5). † *Handbuch*, § 28.

The corresponding problem for $1/\zeta(1+it)$ is, as usual, rather more complicated*. Here we can prove only that

$$\frac{b}{2\beta(1)\vartheta^2} \leq \overline{\lim} \frac{|1/\zeta(1+it)|}{\log\log t} \leq 2\beta(1)\, b,$$

where $b = 6e^{\gamma}\pi^{-2}$, and ϑ is a number arising in the Diophantine approximation, about which all we know is that $\frac{1}{2} \leq \vartheta \leq 2$. This problem would therefore be solved if we could prove that $\beta(1) = \frac{1}{2}$ and $\vartheta = 1$. Actually we can only say that

$$\frac{1}{8} b \leq \overline{\lim} \frac{|1/\zeta(1+it)|}{\log\log t} \leq 2b.$$

5.6. *The function $S(t)$.*

5.61. We begin by proving a formula connecting $\log\zeta(s)$ with the remainder-term in the formula for $N(t)$. We write

$$(1) \qquad N(t) = \frac{1}{2\pi} t \log t - \frac{\log 2\pi e}{2\pi} t + \tfrac{7}{8} + R(t),$$

so that $R(t) = S(t) + O(1/t)$. Thus $R(t)$ and $S(t)$ are not very different, but it is sometimes more convenient to consider the former. Let

$$(2) \qquad \phi(t) = \operatorname*{Max}_{1 \leq u \leq t} |S_1(u)| \qquad (t > 1),$$

so that $\phi(t)$ is positive and non-decreasing for $t > 1$.

THEOREM 60†. *We have*

$$(3) \quad \log\zeta(s) = -\int_{t-x}^{t+x} \frac{R(u)}{u + i(s-\frac{1}{2})} du + O\left\{\frac{\phi(2t)}{x}\right\} + O(1)$$

uniformly for $\sigma > \frac{1}{2}$, $0 < x < \frac{1}{2}t$.

Writing $s = \frac{1}{2} + iz$, where $\mathbf{I}(z) < 0$, we have

$$\tfrac{1}{2} s(s-1)\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s} = \Xi(z) = \Xi(0)\prod_1^\infty\left(1 - \frac{z^2}{\gamma_n^2}\right),$$

where the numbers γ_n are all real, and the number of them less than T is $N(T)$. Now

$$\log\Pi\left(1 - \frac{z^2}{\gamma_n^2}\right) = \Sigma n\left\{\log\left(1 - \frac{z^2}{\gamma_n^2}\right) - \log\left(1 - \frac{z^2}{\gamma_{n+1}^2}\right)\right\}$$

$$= \Sigma n \int_{\gamma_n}^{\gamma_{n+1}} \frac{2z^2}{u(z^2-u^2)} du = \int_{\gamma_1}^{\infty} \frac{2z^2}{u(z^2-u^2)} N(u)\, du.$$

Substituting for $N(u)$ from (1), we obtain a number of integrals which can be evaluated by standard methods, and an integral involving $R(u)$.

* Littlewood (6). † Titchmarsh (3).

The result is

$$\log \Xi(z) = \tfrac{1}{2}iz\log z - \tfrac{1}{4}\pi z - \tfrac{1}{2}iz\log 2\pi e + \tfrac{7}{4}\log z + O(1)$$
$$+ \int_{\gamma_1}^{\infty} \frac{2z^2}{u(z^2-u^2)} R(u)\,du.$$

On the other hand, by the asymptotic formula for $\Gamma(s)$,

$$\log\{\tfrac{1}{2}s(s-1)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\} = \tfrac{1}{2}iz\log z - \tfrac{1}{4}\pi z - \tfrac{1}{2}iz\log 2\pi e + \tfrac{7}{4}\log z + O(1).$$

Hence

$$(4) \qquad \log\zeta(s) = \int_{\gamma_1}^{\infty} \frac{2(s-\frac{1}{2})^2}{u\{u^2+(s-\frac{1}{2})^2\}} R(u)\,du + O(1).$$

Now (using $S_1(u) = O(u)$ for $u > 2t$)

$$\int_{t+x}^{\infty} \frac{S(u)}{u\{u^2+(s-\frac{1}{2})^2\}}\,du$$
$$= -\frac{S_1(t+x)}{(t+x)\{(t+x)^2+(s-\frac{1}{2})^2\}} + \left\{\int_{t+x}^{2t} + \int_{2t}^{\infty}\right\} \frac{3u^2+(s-\frac{1}{2})^2}{u^2\{u^2+(s-\frac{1}{2})^2\}^2} S_1(u)\,du$$
$$= O\left\{\frac{\phi(2t)}{t^2x}\right\} + O\left\{\int_{t+x}^{2t} \frac{\phi(2t)\,du}{t^2(u-t)^2}\right\} + O\left\{\int_{2t}^{\infty} \frac{u\,du}{(u^2-t^2)^2}\right\}$$
$$= O\left\{\frac{\phi(2t)}{t^2x}\right\} + O\left(\frac{1}{t^2}\right).$$

Also

$$\int_{t+x}^{\infty} \frac{R(u)-S(u)}{u\{u^2+(s-\frac{1}{2})^2\}}\,du = O\left\{\int_{t+x}^{\infty} \frac{du}{u^2(u^2-t^2)}\right\} = O\left(\frac{1}{t^2x}\right).$$

The integrals over $(\gamma_1, t-x)$ may be dealt with in the same way. Hence

$$\log\zeta(s) = \int_{t-x}^{t+x} \frac{2(s-\frac{1}{2})^2}{u\{u^2+(s-\frac{1}{2})^2\}} R(u)\,du + O\left\{\frac{\phi(2t)}{x}\right\} + O(1).$$

But

$$\frac{2(s-\frac{1}{2})^2}{u\{u^2+(s-\frac{1}{2})^2\}} = \frac{2}{u} - \frac{1}{u+i(s-\frac{1}{2})} - \frac{1}{u-i(s-\frac{1}{2})}$$

and the results

$$\int_{t-x}^{t+x} \frac{R(u)}{u}\,du = O(1), \qquad \int_{t-x}^{t+x} \frac{R(u)}{u-i(s-\frac{1}{2})}\,du = O(1)$$

are easily obtained by integrating the parts. Hence the result.

5.62. THEOREM 61*. *We have*

$$S(t) = \Omega\{(\log t)^{\frac{1}{2}-\epsilon}\}$$

and

$$S_1(t) = \Omega\{(\log t)^{\frac{1}{2}-\epsilon}\}.$$

* Landau (1), Littlewood (5).

Take $x = \log t$ in Theorem 60. Then, since $\phi(t) = O(\log t)$,

$$\log \zeta(s) = -\int_{t-\log t}^{t+\log t} \frac{R(u)}{u + i(s - \frac{1}{2})} du + O(1).$$

Suppose that $S(t) = O(\log^{\lambda} t)$, i.e. that $R(t) = O(\log^{\lambda} t)$. Then, for fixed $\sigma > \frac{1}{2}$,

$$\log \zeta(s) = O\left\{\log^{\lambda} t \int_{t-\log t}^{t+\log t} \frac{du}{\sqrt{\{(u-t)^2 + (\sigma - \frac{1}{2})^2\}}}\right\} = O(\log^{\lambda} t \log \log t).$$

If $\lambda < \frac{1}{2}$, this contradicts $\nu(\sigma) \geqq 1 - \sigma$ for σ sufficiently near to $\frac{1}{2}$. Hence $\lambda \geqq \frac{1}{2}$, which is the first result.

Again, on integrating by parts, we obtain (for fixed $\sigma > \frac{1}{2}$)

$$\log \zeta(s) = -\int_{t-\log t}^{t+\log t} \frac{S_1(u)}{\{u + i(s - \frac{1}{2})\}^2} du + O(1),$$

and the second result is proved in the same way.

If we define α as the lower bound of numbers λ such that

$$S(t) = O(\log^{\lambda} t),$$

what the above argument really shews is that

$$\alpha \geqq \lim_{\sigma \to \frac{1}{2}} \nu(\sigma).$$

The problem of the order of $S(t)$ is therefore closely connected with the problem of the value of $\nu(\sigma)$.

5.63. It is also possible to prove 'one-sided' results of the above type, viz., that the inequalities*

$$S(t) > (\log t)^{\frac{1}{2}-\epsilon}, \quad S(t) < -(\log t)^{\frac{1}{2}-\epsilon}$$

both have solutions for arbitrarily large values of t.

This result is suggested by what we know about $\nu(\sigma)$. For the two functions $\text{Max}\{\pm \mathbf{I} \log \zeta(s), 0\}$ both have the same ν-function as $\log \zeta(s)$; and from this it follows that, if $\sigma > \frac{1}{2}$, the inequalities

$$\text{am}\,\zeta(s) > (\log t)^{\nu(\sigma)-\epsilon}, \quad \text{am}\,\zeta(s) < -(\log t)^{\nu(\sigma)-\epsilon}$$

both have solutions for arbitrarily large values of t. It is only a matter of shewing that this is still true when σ is replaced by $\frac{1}{2}$.

5.71. THEOREM 62†.

$$S(t) = O\left\{\frac{\log t}{\log \log t}\right\}, \quad S_1(t) = O\left\{\frac{\log t}{(\log \log t)^2}\right\}.$$

The full proof of this is too long to be given here; but we shall indicate the main ideas on which it depends.

* Landau (1), Bohr and Landau (3).

† Landau (6), Cramér (1), Littlewood (4), Titchmarsh (3).

In the first place, it follows in an elementary way from the fact that $N(T)$ is monotonic and $O(\log T)$ that, if $\phi(t)$ is defined by 5.61 (2),

$$(1) \qquad S(t) = O[\sqrt{\{\log t\, \phi(2t)\}}],$$

the occurrence of ϕ to a power less than the first being the important feature of this formula. It follows that the first term on the right of 5.61 (3) depends on $\sqrt{\phi}$, and in fact we obtain

$$\log \zeta(s) = O[x \log\log T \sqrt{\{\log t\, \phi(4t)\}}] + O\{x^{-1}\phi(4t)\} + O(1)$$

for $\sigma - \frac{1}{2} \geqq 1/\log\log T$, $4 \leqq t \leqq T$. We choose x so that the first two terms are of the same order, and obtain

$$(2) \qquad \log \zeta(s) = O[(\log t)^{\frac{1}{4}} (\log\log T)^{\frac{1}{2}} \{\phi(4t)\}^{\frac{3}{4}}].$$

Here ϕ still occurs to a power less than the first.

We next apply Hadamard's three-circles theorem to improve (2), in much the same way that we improved 5.12 (2) to 5.12 (1). The effect of this argument is to divide the right-hand side by a power of $\log T$ which increases with σ; we obtain, roughly,

$$(3) \qquad \log \zeta(s) = O[(\log T)^{\frac{1}{4} - A(\sigma - \frac{1}{2})} (\log\log T)^{\frac{1}{2}} \{\phi(4t)\}^{\frac{3}{4}}].$$

Now by Theorem 39 $S_1(t)$ depends on the integral of $\log|\zeta(s)|$ with respect to σ, and this integration applied to (3) gives us an extra factor $\log\log T$ in the denominator. In fact

$$(4) \qquad \int_{\frac{1}{2}+1/\log\log T}^{2} \log|\zeta(s)|\, d\sigma = O[(\log T)^{\frac{1}{4}} (\log\log T)^{-\frac{1}{2}} \{\phi(4t)\}^{\frac{3}{4}}].$$

We also find that the integral over $(\frac{1}{2}, \frac{1}{2} + 1/\log\log T)$ is of this order. The left-hand side of (4) may therefore be replaced by $S(t)$. If therefore we ignore the difference between t and T, and between the various multiples of t, we obtain

$$\phi(t) = O[(\log t)^{\frac{1}{4}} (\log\log t)^{-\frac{1}{2}} \{\phi(t)\}^{\frac{3}{4}}],$$

which gives the result for $S_1(t)$. The result for $S(t)$ then follows from (1).

5.72. Theorem 62 also enables us to prove inequalities for $\zeta(s)$ in the immediate neighbourhood of $\sigma = \frac{1}{2}$, a region not touched by methods such as that used in Theorem 57.

THEOREM 63*.

$$\zeta(\tfrac{1}{2} + it) = O\left\{\exp\left(A \frac{\log t}{\log\log t}\right)\right\}.$$

Since, by Theorem 62,

$$\phi(t) = O\{\log t/(\log\log t)^2\},$$

* Littlewood (4), Titchmarsh (3).

Theorem 56 gives

$$\log|\zeta(s)| = -\int_{t-x}^{t+x} \frac{u-t}{(u-t)^2+(\sigma-\frac{1}{2})^2} R(u)\,du + O\left\{\frac{\log t}{x(\log\log t)^2}\right\} + O(1)$$

$$= \int_0^x \frac{v}{v^2+(\sigma-\frac{1}{2})^2}\{R(t-v)-R(t+v)\}\,dv + O\left\{\frac{\log t}{x(\log\log t)^2}\right\} + O(1).$$

Now, writing $N(T) = M(T) + R(T)$,

$$R(t-v) - R(t+v) = N(t-v) - N(t+v) + M(t+v) - M(t-v)$$
$$\leqq M(t+v) - M(t-v)$$

(since $N(t)$ is non-decreasing)

$$< Av\log t,$$

since $M'(t) = O(\log t)$. Hence

$$\log|\zeta(s)| < A\int_0^x \frac{v^2\log t}{v^2+(\sigma-\frac{1}{2})^2}\,dv + O\left\{\frac{\log t}{x(\log\log t)^2}\right\} + O(1)$$
$$< Ax\log t + O[\log t/\{x(\log\log t)^2\}] + O(1),$$

and taking $x = 1/\log\log t$ and making $\sigma \to \frac{1}{2}$, the result follows.

5.73. Theorem 64*. *We have*

(1) $\log\zeta(s) = O\left\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\right\} \quad \left(\frac{1}{2} + \frac{1}{\log\log t} \leqq \sigma \leqq 1-\delta\right)$,

(2) $\operatorname{am}\zeta(s) = O\left(\frac{\log t}{\log\log t}\right) \quad \left(\frac{1}{2} \leqq \sigma \leqq \frac{1}{2} + \frac{1}{\log\log t}\right)$,

(3) $-A\frac{\log t}{\log\log t}\log\left\{\frac{2}{(\sigma-\frac{1}{2})\log\log t}\right\} < \log|\zeta(s)| < A\frac{\log t}{\log\log t}$

$$\left(\tfrac{1}{2} < \sigma \leqq \tfrac{1}{2} + \frac{1}{\log\log t}\right).$$

Of these, (1) is an extension of Theorem 57 to a wider range. Using Theorem 62, Theorem 60 gives

$$\log\zeta(s) = O\left(\frac{x}{\sigma-\frac{1}{2}}\frac{\log t}{\log\log t}\right) + O\left\{\frac{\log t}{x(\log\log t)^2}\right\} + O(1),$$

and taking $x = 1/\log\log t$, the result follows for $\sigma = \frac{1}{2} + 1/\log\log t$. It then follows from Theorem 57 in the general case by the Phragmén-Lindelöf method.

Again

$$\operatorname{am}\zeta(s) = \int_0^x \frac{\sigma-\frac{1}{2}}{v^2+(\sigma-\frac{1}{2})^2}\{R(t+v)+R(t-v)\}\,dv + O\left\{\frac{\phi(2t)}{x}\right\} + O(1)$$

$$= O\left\{\frac{\log t}{\log\log t}\int_0^x \frac{\sigma-\frac{1}{2}}{v^2+(\sigma-\frac{1}{2})^2}\,dv\right\} + O\left\{\frac{\log t}{x(\log\log t)^2}\right\} + O(1),$$

* Titchmarsh (3). A result in this direction is given by Cramér (3).

and (2) follows on taking $x=1$. Finally the upper bound (3) follows from § 5.72; and

$$\log|\zeta(s)| = O\left\{\frac{\log t}{\log\log t}\int_0^x \frac{v\,dv}{u^2+(\sigma-\frac{1}{2})^2}\right\} + O\left\{\frac{\log t}{x(\log\log t)^2}\right\} + O(1),$$

which gives the lower bound on taking $x = 1/\log\log t$.

5.8. *Mean value theorems for $S(t)$ and $S_1(t)$**.

The function $\pi S_1(t)$ is, as we know from § 3.72, represented substantially by the integral

$$\int_{\frac{1}{2}}^{2} \log|\zeta(\sigma+it)|\,d\sigma.$$

If we calculate the integral from 5.44 (1), and ignore anything in the nature of an error-term, we obtain

$$\Sigma\, \Lambda_2(n)\, n^{-\frac{1}{2}} \cos(t\log n)\, e^{-\delta n},$$

where $\Lambda_2(n) = \Lambda_1(n)\log n$.

The mean square of this over $(0, T)$ can be calculated after the manner of § 1.23. The result of this somewhat free-handed procedure is

$$\text{(1)} \qquad \int_0^T |S_1(t)|^2\,dt \sim \frac{T}{2\pi^2}\,\Sigma\,\frac{\{\Lambda_2(n)\}^2}{n^2}.$$

This result is true; and similar results hold for integrals of $S(t)$ of higher order, each integration making things easier.

The case of $S(t)$ itself is much more difficult. The series corresponding to that on the right of (1) is $\Sigma\,\{\Lambda_1(n)\}^2/n$, which is divergent; so that we should expect the mean square of $S(t)$ over $(0, T)$ to be of a higher order than T. Actually we can prove that

$$\text{(2)} \qquad \int_0^T |S(t)|^2\,dt > AT\log\log T,$$

but the exact behaviour of this integral remains a mystery. On the other hand we can prove that

$$\text{(3)} \qquad \int_0^T |S(t)|\,dt = O(T\log\log T).$$

The best known upper bound for (2) is obtained from (3) and Theorem 62, and is $AT\log T$.

5.9. *Necessary and sufficient conditions for the truth of the Riemann hypothesis.* We have noticed one such condition in § 5.21, viz. the convergence of $\Sigma\mu(n)\,n^{-s}$ for $\sigma > \frac{1}{2}$. Another necessary and sufficient

* Littlewood (5), Titchmarsh (2).

condition* is that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!\,\zeta(2n)} x^n = O(x^{\frac{1}{4}+\epsilon})$$

as $x \to \infty$.

A condition of quite a different type, which it would take too long to explain here, has been given by Franel†.

CHAPTER VI

LINDELÖF'S HYPOTHESIS

6.11. Lindelöf's hypothesis is that $\zeta(\frac{1}{2}+it) = O(t^\epsilon)$; or, what comes to the same thing, that $\zeta(\sigma+it) = O(t^\epsilon)$ for every $\sigma \geqq \frac{1}{2}$. For either statement is, by the theory of the function $\mu(\sigma)$, equivalent to saying that $\mu(\sigma) = 0$ for $\sigma \geqq \frac{1}{2}$. The hypothesis is suggested by various theorems in Chapters I and II. We have also seen that Lindelöf's hypothesis is true if the Riemann hypothesis is true. The converse deduction however cannot be made. It is therefore of some interest to investigate the consequences of the less drastic hypothesis, since it may be true even if the Riemann hypothesis is false, and in any case might turn out to be much easier to prove.

6.12. THEOREM 65‡. *On Lindelöf's hypothesis*

$$(1) \qquad \int_1^T |\zeta(\sigma+it)|^{2k}\,dt \sim T \sum d_k^2(n)\,n^{-2\sigma}$$

for every positive integer k and $\sigma > \frac{1}{2}$.

This follows at once from Theorem 28, in which we may now take $\lambda = 0$.

It is possible to obtain a similar result for non-integral values of k, but this requires a further argument§.

6.13. THEOREM 66‖. *The converse of Theorem* 65 *is true, i.e. if* 6.12 (1) *holds for every k and $\sigma > \frac{1}{2}$, then Lindelöf's hypothesis is true.*

It is sufficient to assume simply that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T)$$

* Riesz (1), Hardy and Littlewood (2). † Franel (1), Landau (11).
‡ Hardy and Littlewood (5). § Titchmarsh (6).
‖ Hardy and Littlewood (5).

for every k and $\sigma > \frac{1}{2}$. If $\zeta(\sigma+it)$ is not $O(t^\epsilon)$, then there is a positive number λ, and a sequence of numbers $s_\nu = \sigma + it_\nu$, such that $t_\nu \to \infty$ with ν and

$$|\zeta(s_\nu)| > Ct_\nu^{\lambda} \quad (C > 0).$$

On the other hand we know that, for $t \geqq 1$,

$$|\zeta'(s)| < ET^F,$$

E and F being positive absolute constants. Hence

$$|\zeta(\sigma+it) - \zeta(\sigma+it_\nu)| = \left|\int_{t_\nu}^{t} \zeta'(\sigma+iu)\,du\right| < 2E(t-t_\nu)\,t_\nu^F < \tfrac{1}{2}Ct_\nu^\lambda$$

if $|t-t_\nu| \leqq t_\nu^{-F}$, and ν is sufficiently large. Hence

$$|\zeta(\sigma+it)| > \tfrac{1}{2}Ct_\nu^\lambda \quad (|t-t_\nu| \leqq t_\nu^{-F}).$$

Take $T = \frac{2}{3}t_\nu$, so that the interval $(t_\nu - t_\nu^{-F},\ t_\nu + t_\nu^{-F})$ is included in $(T, 2T)$ if ν is large. Then

$$\int_T^{2T} |\zeta(\sigma+it)|^{2k}\,dt > \int_{t_\nu - t_\nu^{-F}}^{t_\nu + t_\nu^{-F}} (\tfrac{1}{2}Ct^\lambda)^{2k}\,dt = 2\,(\tfrac{1}{2}C)^{2k}\,t_\nu^{2k\lambda - F},$$

which is contrary to hypothesis if k is large enough. Hence the result.

6.14. We can obtain a similar result if we are given a mean value formula, not for *all* k but simply for a particular value of k.

It follows from a general theorem on mean values of analytic functions* that if for any value of k

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^{2ka}),$$

then

$$\zeta(\sigma+it) = O(t^{a+1/2k}).$$

For example, if we could prove that

$$\frac{1}{T}\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = O(T^\epsilon),$$

which is true for $k=1$, $k=2$, holds also for $k=4$, we could deduce that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{8}+\epsilon}).$$

6.15. THEOREM 67†. *A necessary and sufficient condition for the truth of Lindelöf's hypothesis is that*

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = O(T^\epsilon)$$

for every k.

The necessity is obvious, and the sufficiency may be proved by the method of § 6.13.

* G. H. Hardy, A. E. Ingham and G. Pólya, "Theorems concerning mean values of analytic fnnctions," *Proc. Royal Soc.* (A) 113 (1927), 542–569 (Theorem 2).

† Hardy and Littlewood (5).

6.21. We next consider the relation between Lindelöf's hypothesis and the distribution of the zeros. The hypothesis has, apparently, no effect on our result concerning the order of $N(\sigma, T)$. But it does give a new result concerning the number of zeros which can occur in a rectangle $T \leqq t < T+1$, $\sigma_0 \leqq \sigma \leqq 1$ $(\sigma_0 > \frac{1}{2})$, and in fact is equivalent to a certain hypothesis about this number.

THEOREM 68*. *A necessary and sufficient condition for the truth of Lindelöf's hypothesis is that for every* $\sigma > \frac{1}{2}$

(1) $$N(\sigma, T+1) - N(\sigma, T) = o(\log T).$$

The necessity of the condition is easily proved. We apply Jensen's formula

$$\log \frac{r^n}{r_1 r_2 \ldots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |f(0)|$$

to the circle with centre $2+it$ and radius $\frac{3}{2} - \frac{1}{4}\delta$, $f(s)$ being $\zeta(s)$. On Lindelöf's hypothesis the right-hand side is less than $o(\log t)$, and, if there are N zeros in the concentric circle with radius $\frac{3}{2} - \frac{1}{2}\delta$, the left-hand side is greater than

$$N \log \{(\tfrac{3}{2} - \tfrac{1}{4}\delta)/(\tfrac{3}{2} - \tfrac{1}{2}\delta)\}.$$

Hence the number of zeros in the circle of radius $\frac{3}{2} - \frac{1}{2}\delta$ is $o(\log t)$; and the result stated, with $\sigma = \frac{1}{2} + \delta$, clearly follows by superposing a number of such circles.

The converse deduction is more difficult. The following proof is due to Littlewood†.

6.22. Let C_1 be the circle with centre $2+iT$ and radius $\frac{3}{2} - \delta$ $(\delta > 0)$, and let Σ_1 denote a summation over zeros of $\zeta(s)$ in C_1. Let C_2 be the concentric circle of radius $\frac{3}{2} - 2\delta$.

Then, for s in C_2,

$$\psi(s) = \frac{\zeta'(s)}{\zeta(s)} - \Sigma_1 \frac{1}{s-\rho} = O\left(\frac{\log T}{\delta}\right).$$

This follows from 1.83 (1), since for each term which is in one of the sums

$$\Sigma_1 \frac{1}{s-\rho}, \qquad \sum_{|t-\gamma|<1} \frac{1}{s-\rho},$$

but not in the other, $|s-\rho| \geqq \delta$; and the number of such terms is

$$O(\log T).$$

* Backlund (4). † Littlewood (4).

Let C_3 be the concentric circle of radius $\frac{3}{2}-3\delta$, C the concentric circle of radius $\frac{1}{2}$. Then $\psi(s)=o(\log T)$ for s in C, since each term is $O(1)$, and by hypothesis the number of terms is $o(\log T)$. Hence Hadamard's three-circles theorem gives, for s in C_3,

$$|\psi(s)|<\{o(\log T)\}^{\alpha}\{O(\log T)/\delta\}^{\beta},$$

where $\alpha+\beta=1$, $0<\beta<1$, α and β depending on δ only. Thus in C_3,

$$\psi(s)=o(\log T),$$

the o depending on δ.

Now

$$\begin{aligned}\int_{\frac{1}{2}+3\delta}^{2}\psi(s)\,d\sigma &= \log\zeta(2+it)-\log\zeta(\tfrac{1}{2}+3\delta+it)\\ &\qquad -\Sigma_1\{\log(2+it-\rho)-\log(\tfrac{1}{2}+3\delta+it-\rho)\}\\ &= O(1)-\log\zeta(\tfrac{1}{2}+3\delta+it)+o(\log T)+\Sigma_1\log(\tfrac{1}{2}+3\delta+it-\rho),\end{aligned}$$

since Σ_1 has $o(\log T)$ terms. Also if $t=T$ the left-hand side is $o(\log T)$. Hence, putting $t=T$ and taking real parts,

$$\log|\zeta(\tfrac{1}{2}+3\delta+iT)|=o(\log T)+\Sigma_1\log|\tfrac{1}{2}+3\delta+iT-\rho|.$$

Since $|\frac{1}{2}+3\delta+iT-\rho|<A$ in C_1, it follows that

$$\log|\zeta(\tfrac{1}{2}+3\delta+iT)|<o(\log T),$$

i.e. that Lindelöf's hypothesis is true.

6.31. THEOREM 69*. *On Lindelöf's hypothesis,*

$$S(t)=o(\log t).$$

The proof is the same as Backlund's proof, given in the Introduction, that (without any hypothesis) $S(t)=O(\log t)$, except that we now use $\zeta(s)=O(t^{\epsilon})$ where we previously used $\zeta(s)=O(t^{A})$.

6.32. THEOREM 70†. *On Lindelöf's hypothesis,*

$$S_1(t)=o(\log t). \tag{1}$$

We have, with our previous notation, as in § 6.22,

$$\log\zeta(\sigma+it)=\Sigma_1\log(\sigma+it-\rho)+o(\log T)\quad(\tfrac{1}{2}+3\delta\leqq\sigma\leqq 2).$$

Hence

$$\int_{\sigma}^{2}\log\zeta(\sigma_1+it)\,d\sigma_1=\Sigma_1\int_{\sigma}^{2}\log(\sigma_1+it-\rho)\,d\sigma_1+o(\log T).$$

Since Σ_1 has $o(\log T)$ terms, and as in § 3.73 each term in the sum is $O(1)$, it follows that

$$\int_{\sigma}^{2}\log\zeta(\sigma_1+it)\,d\sigma_1=o(\log t)\quad(\sigma\geqq\tfrac{1}{2}+3\delta).$$

* Cramér (1), Littlewood (4). † Littlewood (4).

In particular

$$(2) \qquad \int_{\frac{1}{2}+3\delta}^{2} \log|\zeta(\sigma+it)|\,d\sigma = o(\log T).$$

Again, integrating the real part of 3.73 (1) from $\frac{1}{2}$ to $\frac{1}{2}+3\delta$,

$$\int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\zeta(s)|\,d\sigma = \sum_{|\gamma-t|\leqq 1} \int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma + O(\delta\log t).$$

Now

$$\int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma = \tfrac{1}{2}\int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log\{(\sigma-\beta)^2+(\gamma-t)^2\}\,d\sigma < \frac{3\delta}{2}\log(\tfrac{1}{4}+1)$$

and

$$\int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma \geqq \int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\sigma-\beta|\,d\sigma \geqq \int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log\left|\sigma-\tfrac{1}{2}-\frac{3\delta}{2}\right| d\sigma$$
$$= 3\delta\left(\log\frac{3\delta}{2}-1\right).$$

Hence

$$(3) \quad \int_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\zeta(s)|\,d\sigma = \sum_{|\gamma-t|\leqq 1} O(\delta\log 1/\delta) + O(\delta\log t)$$
$$= O\{\delta\log(1/\delta)\log t\}.$$

Combining (2) and (3) with Theorem 39, we have

$$S_1(t) = \int_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma + O(1)$$
$$= O\{\delta\log(1/\delta)\log t\} + o(\log t) + O(1),$$

and the result follows.

APPENDIX

A proof of Kronecker's theorem.

The following proof of Kronecker's theorem (§ 1.26) is due to Bohr, *Proc. London Math. Soc.* (2) 21 (1922), 315–316. It depends on ideas somewhat similar to those used in proving Theorem 3.

It is clearly sufficient to prove that we can find a number t such that each of the numbers

$$e^{2\pi i (a_n t - b_n)} \qquad (n = 1, 2, \dots N)$$

differs from 1 by less than ϵ; or, if

$$F(t) = 1 + \sum_{n=1}^{N} e^{2\pi i (a_n t - b_n)},$$

that the upper bound of $|F(t)|$ for real values of t is $N+1$. Let us denote this upper bound by L.

Let
$$G(\phi_1, \phi_2, \dots, \phi_N) = 1 + \sum_{n=1}^{N} e^{2\pi i \phi_n},$$

where the numbers $\phi_1, \phi_2, \dots, \phi_N$ are independent real variables, each lying in the interval (0, 1). Then the upper bound of $|G|$ is $N+1$, this being the value of $|G|$ when $\phi_1 = \phi_2 = \dots = \phi_N = 0$.

We consider the polynomial expansions of $\{F(t)\}^k$ and $\{G(\phi_1, \dots, \phi_N)\}^k$, where k is an arbitrary positive integer; and we observe that each of these expansions contains the same number of terms. For, the numbers $a_1, a_2, \dots, a_N$ being linearly independent, no two terms in the expansion of $\{F(t)\}^k$ fall together. Also the moduli of corresponding terms are equal. Hence, by an obvious extension of the lemma of § 1.23, the two mean values

$$F_k = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\{F(t)\}^k|^2 \, dt,$$

$$G_k = \int_0^1 \int_0^1 \dots \int_0^1 |\{G(\phi_1, \phi_2, \dots, \phi_N)\}^k|^2 \, d\phi_1 \, d\phi_2 \dots d\phi_N$$

are equal, their common value being the sum of the squares of the moduli of the terms in the polynomial expansions.

Since the formula of § 1.22 can clearly be extended to an N-fold integral, we have

$$\lim_{k\to\infty} G_k^{1/2k} = N + 1.$$

Hence also
$$\lim_{k\to\infty} F_k^{1/2k} = N + 1.$$

But plainly
$$F_k^{1/2k} \leqq L$$

for all values of k. Hence $L \geqq N + 1$, which proves the theorem.

BIBLIOGRAPHY

[This is a list of the principal memoirs on the zeta-function which have appeared since Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, 1909. An exhaustive list of previous memoirs is given in Landau's work.]

K. ANANDA-RAU.

(1) The infinite product for $(s-1)\zeta(s)$, *Math. Zeitschrift* 20 (1924), 156–164.

R. BACKLUND.

(1) Einige numerische Rechnungen, die Nullpunkte der Riemannschen ζ-Funktion betreffend, *Öfversigt Finska Vetensk. Soc.* (A) 54 (1911–12), No. 3.

(2) Sur les zéros de la fonction $\zeta(s)$ de Riemann, *Comptes Rendus* 158 (1914), 1979–1981.

(3) Über die Nullstellen der Riemannschen Zetafunktion, *Acta Math.* 41 (1918), 345–375.

(4) Über die Beziehung zwischen Anwachsen und Nullstellen der Zetafunktion, *Öfversigt Finska Vetensk. Soc.* 61 (1918–19), No. 9.

H. BOHR.

(1) Über das Verhalten von $\zeta(s)$ in der Halbebene $\sigma > 1$, *Göttinger Nachrichten* (1911), 409–428.

(2) Sur l'existence de valeurs arbitrairement petites de la fonction $\zeta(s) = \zeta(\sigma+it)$ de Riemann pour $\sigma > 1$, *Oversigt Vidensk. Selsk. København* (1911), 201–208.

(3) Sur la fonction $\zeta(s)$ dans le demi-plan $\sigma > 1$, *Comptes Rendus* 154 (1912) 1078–1081.

(4) Über die Funktion $\zeta'(s)/\zeta(s)$, *Journal für Math.* 141 (1912), 217–234.

(5) Note sur la fonction zéta de Riemann $\zeta(s) = \zeta(\sigma+it)$ sur la droite $\sigma = 1$, *Oversigt Vidensk. Selsk. København* (1913), 3–11.

(6) Lösung des absoluten Konvergenzproblems einer allgemeinen Klasse Dirichletscher Reihen, *Acta Math.* 36 (1913), 197–240.

(7) Sur la fonction $\zeta(s)$ de Riemann, *Comptes Rendus* 158 (1914), 1986–1988.

(8) Zur Theorie der Riemannschen Zetafunktion im kritischen Streifen, *Acta Math.* 40 (1915), 67–100.

H. BOHR and R. COURANT.

(1) Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion, *Journal für Math.* 144 (1914), 249–274.

H. BOHR and E. LANDAU.

(1) Über das Verhalten von $\zeta(s)$ und $\zeta_{\mathfrak{K}}(s)$ in der Nähe der Geraden $\sigma = 1$, *Göttinger Nachrichten* (1910), 303–330.

(2) Über die Zetafunktion, *Rend. di Palermo* 32 (1911), 278–285.
(3) Beiträge zur Theorie der Riemannschen Zetafunktion, *Math. Annalen* 74 (1913), 3–30.
(4) Ein Satz über Dirichletsche Reihen mit Anwendung auf die ζ-Funktion und die L-Funktionen, *Rend. di Palermo* 37 (1914), 269–272.
(5) Sur les zéros de la fonction $\zeta(s)$ de Riemann, *Comptes Rendus* 158 (1914), 106–110.
(6) Über das Verhalten von $1/\zeta(s)$ auf der Geraden $\sigma=1$, *Göttinger Nachrichten* (1923), 71–80.
(7) Nachtrag zu unseren Abhandlungen aus den Jahrgängen 1910 und 1923, *Göttinger Nachrichten* (1924), 168–172.

H. Bohr, E. Landau and J. E. Littlewood.

(1) Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma=\frac{1}{2}$, *Bull. Acad. Belgique* 15 (1913), 1144–1175.

C. Burrau.

(1) Numerische Lösung der Gleichung $\dfrac{2^{-D}\log 2}{1-2^{-2D}}=\sum\limits_{n=2}^{\infty}\dfrac{p_n{}^{-D}\log p_n}{1-p_n{}^{-2D}}$ wo p_n die Reihe der Primzahlen von 3 an durchläuft, *Journal für Math.* 142 (1912), 51–53.

F. Carlson.

(1) Über die Nullstellen der Dirichletschen Reihen und der Riemannschen ζ-Funktion, *Arkiv för Mat. Astr. och Fysik* 15 (1920), No. 20.
(2), (3) Contributions à la théorie des séries de Dirichlet, *Arkiv för Mat. Astr. och Fysik* 16 (1922), No. 18, and 19 (1926), No. 25.

H. Cramér.

(1) Über die Nullstellen der Zetafunktion, *Math. Zeitschrift* 2 (1918), 237–241.
(2) Studien über die Nullstellen der Riemannschen Zetafunktion, *Math. Zeitschrift* 4 (1919), 104–130.
(3) Bemerkung zu der vorstehenden Arbeit des Herrn E. Landau, *Math. Zeitschrift* 6 (1920), 155–157.

H. Cramér and E. Landau.

(1) Über die Zetafunktion auf der Mittellinie des kritischen Streifens, *Arkiv för Mat. Astr. och Fysik* 15 (1920), No. 28.

M. Fekete.

(1) The zeros of Riemann's zeta-function on the critical line, *Journal London Math. Soc.* 1 (1926), 15–19.

J. Franel.

(1) Les suites de Farey et le problème des nombres premiers, *Göttinger Nachrichten* (1924), 198–201.

T. Gronwall.

(1) Sur la fonction $\zeta(s)$ de Riemann au voisinage de $\sigma=1$, *Rend. di Palermo* 35 (1913), 95–102.

H. Hamburger.

(1), (2), (3) Über die Riemannsche Funktionalgleichung der ζ-Funktion, *Math. Zeitschrift* 10 (1921), 240–254; 11 (1922), 224–245; 13 (1922), 283–311.

(4) Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen ζ-Funktion äquivalent sind, *Math. Annalen* 85 (1922), 129–140.

G. H. Hardy.

(1) Sur les zéros de la fonction $\zeta(s)$ de Riemann, *Comptes Rendus* 158 (1914), 1012–1014.

(2) On the integration of Fourier series, *Messenger of Math.* 51 (1922), 186–192.

(3) A new proof of the functional equation for the zeta-function, *Mat. Tidsskrift* B (1922), 71–73.

(4) Note on a theorem of Mertens, *Journal London Math. Soc.* 2 (1926), 70–72.

G. H. Hardy and J. E. Littlewood.

(1) Some problems of Diophantine approximation, *Internat. Congress of Math., Cambridge* (1912), 1, 223–229.

(2) Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, *Acta Math.* 41 (1918), 119–196.

(3) The zeros of Riemann's zeta-function on the critical line, *Math. Zeitschrift* 10 (1921), 283–317.

(4) The approximate functional equation in the theory of the zeta-function, with applications to the divisor problems of Dirichlet and Piltz, *Proc. London Math. Soc.* (2) 21 (1922), 39–74.

(5) On Lindelöf's hypothesis concerning the Riemann zeta-function, *Proc. Royal Soc.* (A) 103 (1923), 403–412.

(6) The approximate functional equations for $\zeta(s)$ and $\zeta^2(s)$, *Proc. London Math. Soc.* (2) 29 (1929), 81–97.

E. Hecke.

(1) Über die Lösungen der Riemannschen Funktionalgleichung, *Math. Zeitschrift* 16 (1923), 301–307.

G. Hoheisel.

(1) Über das Verhalten des reziproken Wertes der Riemannschen Zeta-Funktion, *Sitzungsber. Preuss. Akad. Wiss.* (1929), 219–223.

J. I. Hutchinson.

(1) On the roots of the Riemann zeta-function, *Trans. Amer. Math. Soc.* 27 (1925), 49–60.

A. E. INGHAM.

(1) Mean-value Theorems in the theory of the Riemann zeta-function, *Proc. London Math. Soc.* (2) 27 (1926), 273–300.

H. D. KLOOSTERMAN.

(1) Een integraal voor de ζ-functie van Riemann, *Christiaan Huygens Math. Tijdschrift* 2 (1922), 172–177.

E. LANDAU.

(1) Zur Theorie der Riemannschen Zetafunktion, *Vierteljahrsschr. Naturf. Ges. Zürich* 56 (1911), 125–148.

(2) Über die Nullstellen der Zetafunktion, *Math. Annalen* 71 (1911), 548–564.

(3) Über einige Summen, die von den Nullstellen der Riemannschen Zetafunktion abhangen, *Acta Math.* 35 (1912), 271–294.

(4) Über die Hardysche Entdeckung unendlich vieler Nullstellen der Zetafunktion mit reellem Teil $\frac{1}{2}$, *Math. Annalen* 76 (1915), 212–243.

(5) Neuer Beweis eines Satzes von Herrn Valiron, *Jahresbericht der Deutschen Math. Verein* 29 (1920), 239.

(6) Über die Nullstellen der Zetafunktion, *Math. Zeitschrift* 6 (1920), 151–154.

(7) Über die Nullstellen der Dirichletschen Reihen und der Riemannschen ζ-Funktion, *Arkiv för Mat. Astr. och Fysik* 16 (1921), No. 7.

(8) Über die Möbiussche Funktion, *Rend. di Palermo* 48 (1924), 277–280.

(9) Über die Wurzeln der Zetafunktion, *Math. Zeitschrift* 20 (1924), 98–104.

(10) Über die ζ-Funktion und die L-Funktionen, *Math. Zeitschrift* 20 (1924), 105–125.

(11) Bemerkung zu der vorstehenden Arbeit von Herrn Franel, *Göttinger Nachrichten* (1924), 202–206.

(12) Über die Riemannsche Zetafunktion in der Nähe von $\sigma=1$, *Rend. di Palermo* 50 (1926), 423–427.

(13) Über die Zetafunktion und die Hadamardsche Theorie der ganzen Funktionen, *Math. Zeitschrift* 26 (1927), 170–175.

(14) Bemerkung zu einer Arbeit von Hrn. Hoheisel über die Zetafunktion, *Sitzungsber. Preuss. Akad. Wiss.* (1929), 271–275.

J. E. LITTLEWOOD.

(1) Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $\mathbf{R}(s) > \frac{1}{2}$, *Comptes Rendus* 154 (1912), 263–266.

(2) Researches in the theory of the Riemann ζ-function, *Proc. London Math. Soc.* (2) 20 (1922), Records XXII–XXVIII.

(3) Two notes on the Riemann zeta-function, *Proc. Camb. Phil. Soc.* 22 (1924), 234–242.

(4) On the zeros of the Riemann zeta-function, *Proc. Camb. Phil. Soc.* 22 (1924), 295–318.

(5) On the Riemann zeta-function, *Proc. London Math. Soc.* (2) 24 (1925), 175–201.

(6) On the function $1/\zeta(1+ti)$, *Proc. London Math. Soc.* (2) 27 (1928), 349-357.

R. MATTSON.

(1) Eine neue Darstellung der Riemann'schen Zetafunktion, *Arkiv för Mat. Astr. och Fysik* 19 (1926), No. 26.

Hj. MELLIN.

(1) Über die Nullstellen der Zetafunktion, *Annales Acad. Scientiarum Fennicae* (A) 10 (1917), No. 11.

L. J. MORDELL.

(1) Some applications of Fourier Series in the analytic theory of numbers, *Proc. Camb. Phil. Soc.* 34 (1928), 585–596.

(2) Poisson's summation formula and the Riemann zeta-function, *Journal London Math. Soc.* 4 (1929), 285–291.

C. H. MÜNTZ.

(1) Beziehungen der Riemannschen ζ-Funktion zu willkürlichen reellen Funktionen, *Mat. Tidsskrift* B (1922), 39–47.

F. and R. NEVANLINNA.

(1), (2) Über die Nullstellen der Riemannschen Zetafunktion, *Math. Zeitschrift* 20 (1924), 253–263; and 23 (1925), 159–160.

G. PÓLYA.

(1) Bemerkung über die Integraldarstellung der Riemannschen ξ-Funktion, *Acta Math.* 48 (1926), 305–317.

(2) On the zeros of certain trigonometric integrals, *Journal London Math. Soc.* 1 (1926), 98–99.

(3) Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen, *Kgl. Danske Videnskabernes Selskab.* 7 (1927), No. 17.

(4) Über trigonometrische Integrale mit nur reellen Nullstellen, *Journal für Math.* 158 (1927), 6–18.

S. RAMANUJAN.

(1) New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$, *Quart. J. of Math.* 46 (1915), 253-261.

(2) Some formulae in the analytic theory of numbers, *Messenger of Math.* 45 (1915), 81-84.

M. RIESZ.

(1) Sur l'hypothèse de Riemann, *Acta Math.* 40 (1916), 185–190.

C. SIEGEL.

(1) Bemerkung zu einem Satz von Hamburger über die Funktionalgleichung der Riemannschen Zetafunktion, *Math. Annalen* 86 (1922), 276–279.

R. v. STERNECK.

(1) Neue empirische Daten über die zahlentheoretische Funktion $\sigma(n)$, *Internat. Congress of Math., Cambridge* (1912), 1, 341–343.

E. C. TITCHMARSH.

(1) The mean-value of the zeta-function on the critical line, *Proc. London Math. Soc.* (2) 27 (1928), 137–150.

(2) On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$, *Proc. London Math. Soc.* (2) 27 (1928), 449–458.

(3) A consequence of the Riemann hypothesis, *Journal London Math. Soc.* 2 (1927), 247–254.

(4) On an inequality satisfied by the zeta-function of Riemann, *Proc. London Math. Soc.* (2) 28 (1928), 70–80.

(5) On the zeros of the Riemann zeta-function, *Proc. London Math. Soc.* (2) 30 (1929), 319–321.

(6) Mean value theorems in the theory of the Riemann zeta-function. *Messenger of Math.* 58 (1929), 125–129.

G. VALIRON.

(1) Sur les fonctions entières d'ordre nul et d'ordre fini, *Thèse*, Paris (1914), and *Annales de Toulouse* (3) 5 (1914), 117–257.

Ch. de la VALLÉE-POUSSIN.

(1), (2) Sur les zéros de $\zeta(s)$ de Riemann, *Comptes Rendus* 163 (1916), 418–421, and 471–473.

A. WALFISZ.

(1) Zur Abschätzung von $\zeta(\frac{1}{2}+it)$, *Göttinger Nachrichten* (1924), 155–158.

A. WALTHER.

(1) Anschauliches zur Riemannschen Zetafunktion, *Acta Math.* 48 (1926), 393–400.

H. WEYL.

(1) Zur Abschätzung von $\zeta(1+it)$, *Math. Zeitschrift* 10 (1921), 88–101.

S. WIGERT.

(1) Sur la théorie de la fonction $\zeta(s)$ de Riemann, *Arkiv för Mat. Astr. och Fysik* (1919), No. 12.

(2) On a problem concerning the Riemann ζ-function, *Proc. Camb. Phil. Soc.* 21 (1921), 17–21.

B. M. WILSON.

(1) Proofs of some formulae enunciated by Ramanujan, *Proc. London Math. Soc.* (2) 21 (1922), 235–255.

J. R. WILTON.

(1) Note on the zeros of Riemann's ζ-function, *Messenger of Math.* 45 (1915), 180–183.

(2) A proof of Burnside's formula for $\log \Gamma(x+1)$ and certain allied properties of Riemann's ζ-function, *Messenger of Math.* 52 (1922), 90–93.

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